Modern Classical Homotopy Theory

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Chapter 1

Categories and Functors

The subject of algebraic topology is historically one of the first in which huge diagrams of functions became a standard feature. The language of category theory is intended to provide tools for understanding such diagrams, for working with them, and for studying the relations between them.

1.1 Diagrams

Before getting to categories, let's engage in an informal discussion of diagrams. Roughly speaking, a **diagram** is a collection (possibly infinite) of 'objects' denoted A, B, X, Y, etc., and (labelled) 'arrows' between the objects, as in the examples

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "h"'] & B \arrow[d, "g"] \\
& C
\end{tikzcd}$$

and

$$\begin{tikzcd}
X_1 \arrow[rr, "f_{12}"] \arrow[dr, "f_{13}"] \arrow[dd, "h_1"'] & & X_2 \arrow[dr, "f_{24}"] \arrow[dd, "h_2" near start] & \\
& X_3 \arrow[rr, crossing over, "f_{34}" near start] & & X_4 \arrow[dd, "h_4"] \\
Y_1 \arrow[rr, "g_{12}" near end] \arrow[dr, "g_{13}"] & & Y_2 \arrow[dr, "g_{24}"] & \\
& Y_3 \arrow[rr, "g_{34}"] \arrow[from=uu, crossing over, "h_3" near start] & & Y_4 .
\end{tikzcd}$$

Each arrow has a **domain** and a **target** – thus $X \xrightarrow{f} Y$ is a simple diagram with a single arrow f whose domain is X and whose target is Y. If g is another arrow with domain Y, then we can form the 'composite arrow' $g \circ f$

with domain X and target Y. The triangle

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\
 & Z
\end{tikzcd}$$

is **commutative** if $h = g \circ f$. If the diagram above is commutative, then we say that h **factors through** f and through g; we also say that h factors through Y. If X and Y are objects in a diagram, we may be able to use the various arrows and their composites to obtain many potentially distinct arrows from X to Y; for example, in the cube diagram, there are precisely 6 different composites from X_1 to Y_4. Each of these paths represents an arrow $X_1 \to Y_4$, but it can happen that different paths become, on composition, the same arrow. If it turns out that, for each pair X, Y of objects in the diagram, all of the possible composite paths from X to Y are ultimately *same* arrow, then we say that the diagram is **commutative**.

We can expand a given (not necessarily commutative) diagram $\mathcal{D}$ by drawing as arrows all of the composites of the given arrows, as well as 'identity arrows' from each 'vertex object' to itself, which compose like identity maps. We'll refer to the expanded diagram as $\overline{\mathcal{D}}$.

Exercise 1.1 Show that $\mathcal{D}$ is commutative if and only if $\overline{\mathcal{D}}$ is commutative.

It is frequently helpful to express complicated definitions and properties in terms of diagrams.

Here's an example. Let F be a field, and let $F \subseteq E$ be a field extension. Then there is an **inclusion map** $f : F \to E$, which just carries an element $\alpha \in F$ to the same element, but thought of as being an element of E. Now an **algebraic closure** for F is an algebraic field extension $a : F \to A$ such that for any other algebraic extension f, there is a unique map $\bar{f}$ making the diagram

$$\begin{tikzcd}
F \arrow[r, "a"] \arrow[d, "f"'] & A \\
E \arrow[ur, dotted, bend right, "\exists!\ \bar{f}"'] &
\end{tikzcd}$$

commutative. Here you should observe that we use dotted arrows to denote arrows that we do not know exist. Also, this definition gives the algebraic closure as a solution to a 'universal problem.'

Discuss universal problems further....

Exercise 1.2

(a) Take some time to convince yourself that the given definition of algebraic closure actually does define what you think of as algebraic closure.

(b) The isomorphism theorems of elementary group theory can be written down in terms of diagrams. Do it!

(c) Rewrite the statement that the cube-shaped diagram above is commutative without using any diagrams at all. [1]

1.2 Categories

Informally, a category is simply a 'complete' list of all the *things* you plan to study together with a complete list of all the *allowable maps* between those things. So an algebraist might work in the category of groups and homomorphisms, while a topologist might work in the category of topological spaces and continuous functions, and a geometer could work in the category of subsets of the plane and rigid motions.

Here's the formal definition.

Definition 1 A **category** $\mathcal{C}$ consists of two things: a collection[2] $\mathrm{ob}(\mathcal{C})$, called the **objects** of $\mathcal{C}$, and, for each $X, Y \in \mathrm{ob}(\mathcal{C})$, a *set* $\mathrm{mor}_{\mathcal{C}}(X, Y)$, called the set of **morphisms** from X to Y. These are subject to the following conditions:

1. If $X, Y, Z \in \mathrm{ob}(\mathcal{C})$, $f \in \mathrm{mor}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{mor}_{\mathcal{C}}(Y, Z)$, then there is another morphism $g \circ f \in \mathrm{mor}_{\mathcal{C}}(X, Z)$ (which you should think of as the composite of f and g).[3]

2. The composition operation is associative: $f \circ (g \circ h) = (f \circ g) \circ h$. Diagramatically, this says that the diagram

$$\begin{tikzcd}
W \arrow[r, "h"] \arrow[dr, "g\circ h"'] & X \arrow[d, "g"] \arrow[dr, "f\circ g"] & \\
& Y \arrow[r, "f"'] & Z
\end{tikzcd}$$

is commutative.

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[1] Thanks to Jason Trowbridge for this idea.

[2] The vague word 'collection' is intended to gloss over some technical set-theoretical issues here. The idea is that the collection of objects should be allowed to be larger than any set, so we can't call it a set of objects. Many authors use a **class** of objects, which is a well defined concept in set theory (or logic); but one of the go-to books on category theory (Mac Lane) uses a set-theoretical trick to get around classes.

[3] This rule makes it meaningful to ask whether a given diagram of objects and arrows in $\mathcal{C}$ is commutative or not.

3. For each $X \in \mathrm{ob}(\mathcal{C})$, there is a special morphism $\mathrm{id}_X \in \mathrm{mor}_{\mathcal{C}}(X, X)$ which satisfies $\mathrm{id}_X \circ f = f$ for any $f \in \mathrm{mor}_{\mathcal{C}}(W, X)$ and $g \circ \mathrm{id}_X = g$ for any $g \in \mathrm{mor}_{\mathcal{C}}(X, Y)$. In diagram form:

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[dr, "f"'] & X \arrow[d, "\mathrm{id}_X"] \arrow[dr, "g"] & \\
& X \arrow[r, "g"'] & Y.
\end{tikzcd}$$

Here are some simple examples to think about.

(a) The category whose objects are groups, and whose morphisms are group homomorphisms.

(b) The category whose objects are topological spaces, and whose morphisms are continuous functions.

(c) The category whose objects are the numbers $1, 2, 3, \ldots$, and such that there is a unique morphism $n \to m$ if n divides m, and no morphisms $n \to m$ if n does not divide m.

(d) The category whose objects are the real numbers, and such that there is a unique morphism $x \to y$ if $x \leq y$, and no morphism if $x > y$.

There is a lot of shorthand that is often used when confusion is unlikely. For example, we usually write $X \in \mathcal{C}$ instead of $X \in \mathrm{ob}(\mathcal{C})$; and rather than $f \in \mathrm{mor}_{\mathcal{C}}(X, Y)$, we write $f : X \to Y$.

EXERCISE 1.3

(a) Give five examples of categories besides the ones already mentioned.

(b) Find a way to interpret a group G as a category with a single object.

(c) Let X be a topological space. Show how to make a category whose objects are the points of X, and such that the set of morphisms from a to b is the set of all paths $\omega : [0, d] \to X$ (where $d \geq 0$) such that $\omega(0) = a$ and $\omega(d) = b$.

(d) Suppose $\mathcal{D}$ is a diagram in the sense of Section 1.1. Show that $\overline{\mathcal{D}}$ is a category.

Lots of basic mathematical ideas are 'best' expressed in the language of categories. For example: a morphism $f : X \to Y$ is an **equivalence** if there is a morphism $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. It is the usual practice to write $g = f^{-1}$ in this case.

EXERCISE 1.4 Show that if such a g exists, it is unique.

PROBLEM 1.5 Let's say $X \sim Y$ if there is an equivalence $f : X \to Y$.

(a) Show that $\sim$ is an equivalence relation.

(b) Interpret 'equivalence' in each of the categories that have been discussed in the text so far, including the ones you found in Exercise 1.3.

PROBLEM 1.6 Let $\mathcal{C}$ be a category and let $f : X \to Y$ be a map which has a left inverse $g : Y \to X$, and suppose g also has a left inverse. Show that f and g are two-sided inverses of each other.

Another simple – but extremely useful – idea is that of a retract. If $A, X \in \mathcal{C}$, then A is a **retract** of X if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"'] \arrow[rr, bend left, "\mathrm{id}_A"] & X \arrow[r, "r"'] & A
\end{tikzcd}$$

If $f : A \to B$ and $g : X \to Y$, then f is a retract of g if there is a commutative diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[rr, bend left, "\mathrm{id}_A"] \arrow[d, "f"'] & X \arrow[r, "r"] \arrow[d, "g"] & A \arrow[d, "f"] \\
B \arrow[r, "j"] \arrow[rr, bend right, "\mathrm{id}_B"'] & Y \arrow[r, "s"] & B
\end{tikzcd}$$

EXERCISE 1.7 Whenever we use the term 'retract' we should be referring to the definition above, where an object A was a retract of another object X in some category. By setting up an appropriate category, show that our definition of f being a retract of g can be thought of as an instance of that general categorical definition. What is does it mean, in terms of the category $\mathcal{C}$ for two objects to be equivalent in your new category?

HINT Obviously, f and g must be among the objects in your category!

EXERCISE 1.8 Find examples of retracts in algebra, topology, and other contexts.

PROBLEM 1.9 Let $f : A \to B$ and $g : X \to Y$ be morphisms in a category $\mathcal{C}$. Assume that f is a retract of g.

(a) Show that if g is an equivalence, then f is also an equivalence.

(b) Show by example that f can be an equivalence even if g is not an equivalence.

1.3 Functors

As you have no doubt experienced, it seldom happens that any serious mathematical study is performed entirely inside a single category. For example, when Galois set out to study fields, he was forced to also work in the category of groups; it was the relationship between these two categories that led to new insights. Another algebraic example is given by group actions, in which the category of groups is studied using the category of sets; playing these two categories off of one another is how the Sylow theorems are generally proved.

A **functor** is a formalism for comparing categories; you can think of a functor intuitively as a morphism from one category to another one. Just as a homomorphism respects the algebraic structure of a group, and a continuous map respects the topological structure of a space, a functor must respect the key features of categories. Functors take objects to objects and morphisms to morphisms; they respect composition, and preserve identity morphisms.

There are actually two kinds of functors – those which reverse the direction of morphisms, and those which don't.

Definition 2 A **covariant functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a function

$$F : \mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$$

and, for each $X, Y \in \mathrm{ob}(\mathcal{C})$, a function

$$F : \mathrm{mor}_{\mathcal{C}}(X, Y) \to \mathrm{mor}_{\mathcal{D}}(F(X), F(Y)).$$

These must satisfy the following conditions:

(a) $F(g \circ f) = F(g) \circ F(f)$

(b) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in \mathrm{ob}(\mathcal{C})$.

Notice that if $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor and $f : X \to Y$ is a morphism in $\mathcal{C}$, then $F(f) : F(X) \to F(Y)$; thus, F carries the domain of f to the domain of $F(f)$, and similarly for the targets. In other words, $F(f)$ points in 'the same direction' as f. This is the meaning of the word 'covariant.'

EXERCISE 1.10

(a) Let $\mathcal{D}$ be a diagram in a category $\mathcal{A}$. Show that $\mathcal{D}$ is commutative if and only if for any two objects $X, Y \in \mathcal{D}$, $\mathrm{mor}_{\mathcal{D}}(X, Y)$ has at most one element.

(b) Show that if you apply a covariant functor $F : \mathcal{A} \to \mathcal{B}$ to a commutative diagram in $\mathcal{A}$, the result is a commutative diagram in $\mathcal{B}$.

PROBLEM 1.11 Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Show that if $f : X \to Y$ is an equivalence in $\mathcal{C}$, then $F(f)$ is an equivalence in $\mathcal{D}$. Is it possible that $F(f)$ is an equivalence without f being an equivalence?

EXERCISE 1.12 Let G be a group, and think of it as a category with one morphism, as in Exercise 1.3. Give an interpretation for functors $F : G \to \text{Sets}$ in terms of familiar concepts in algebra.

Let's use these ideas to prove something topological.[4]

PROBLEM 1.13 Let $i : S^1 \hookrightarrow D^2$ be the inclusion of the circle into the disk. An early success of algebraic topology concerned the existence of a continuous function $r : D^2 \to S^1$ such that the composite $r \circ i : S^1 \to S^1$ is the identity (in other words: is S^1 a retract of D^2?). The fundamental group is a covariant functor

$$\pi_1 : \text{Topological Spaces} \to \text{Groups}$$

such that $\pi_1(S^1) = \mathbb{Z}$ and $\pi_1(D^2) = 0$. Using this functor, show that there can be no such function r.

The second kind of functor is just the same, except that it reverses the direction of arrows.

Definition 3 A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ consists of a function

$$F : \text{ob}(\mathcal{C}) \to \text{ob}(\mathcal{D})$$

and, for each $X, Y \in \text{ob}(\mathcal{C})$, a function

$$F : \text{mor}_{\mathcal{C}}(X, Y) \to \text{mor}_{\mathcal{D}}(F(Y), F(X)).$$

These must satisfy the following conditions:

(a) $F(g \circ f) = F(f) \circ F(g)$

(b) $F(\text{id}_X) = \text{id}_{F(X)}$ for any $X \in \text{ob}(\mathcal{C})$.

In contrast to the covariant functors, if $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor and $f : X \to Y$ in $\mathcal{C}$, then $F(f) : F(Y) \to F(X)$. Thus a contravariant functor carries the domain of f to the target of $F(f)$, and carries the target of f to the domain of $F(f)$ – it 'reverses the direction' of arrows.

EXERCISE 1.14 Let $F : \mathcal{C} \to \mathcal{D}$ be a contravariant functor.

(a) Show that if you apply F to a commutative diagram in $\mathcal{C}$, the result is a commutative diagram in $\mathcal{D}$.

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[4] i.e., interesting!

(b) Show that if $f : X \to Y$ is an equivalence in $\mathcal{C}$, then $F(f)$ is an equivalence in $\mathcal{D}$.

Exercise 1.15 Show that the composite of two functors is a functor. What happens if one or both of the functors is contravariant?

Exercise 1.16 Can there be a functor $F : \mathcal{C} \to \mathcal{D}$ that is *both* covariant and contravariant? What special properties must such a functor have?

Exercise 1.17 Show that there is a universal example for contravariant functors out of a category $\mathcal{C}$. That is, show that there is a category $\mathcal{C}^{\mathrm{op}}$ and a contravariant functor $C \to \mathcal{C}^{\mathrm{op}}$ so that every other contravariant functor $\mathcal{C} \to \mathcal{D}$ has a unique factorization $\mathcal{C} \to \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$, where the functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is covariant.

The category $\mathcal{C}^{\mathrm{op}}$ is known as the **opposite category** of $\mathcal{C}$. Some authors choose not to use contravariant functors at all, and instead use covariant functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Let's look at some simple functors.

Exercise 1.18 Consider the categories **Ab** of abelian groups (and homomorphisms), and **Sets** of sets (and functions). Since an abelian group is a set together with extra structure, we can define $F : \mathbf{Ab} \to \mathbf{Sets}$ by

$$G \mapsto [G, \text{ but completely forgetting the group structure}].$$

Complete the definition of F on morphisms, and show that F is a functor.

Any functor of this kind, in which the target category is a dumbed-down version of the domain, and the functor consists of just getting dumber, is called a **forgetful functor**.

Exercise 1.19 Try to make a formal definition of 'forgetful functor.'

Exercise 1.20 Let $\mathcal{V}$ denote the category of all vector spaces (over the real numbers, say) and all linear transformations. Thus

$$\mathrm{mor}_{\mathcal{V}}(V, W) = \mathrm{Hom}(V, W) = \{T : V \to W \,|\, T \text{ is } \mathbb{R}\text{−linear}\}.$$

(a) Define $F : \mathcal{V} \to \mathcal{V}$ by the rules

$$F(V) = \mathrm{Hom}(\mathbb{R}, V) \quad \text{and} \quad F(f) : g \mapsto f \circ g$$

Show that F is a covariant functor.

(b) Define $G : \mathcal{V} \to \mathcal{V}$ by the rules

$$G(V) = \mathrm{Hom}(V, \mathbb{R}) \quad \text{and} \quad G(f) : g \mapsto g \circ f.$$

Show that G is a contravariant functor.

The functors described in Exercise 1.20 are specific examples of what are, for us, the two most important general kinds of functors.

Proposition 4 *Let $\mathcal{C}$ be a category, and let $A, B \in \mathrm{ob}(\mathcal{C})$.*

- (a) *For $f : X \to Y$, write $f^* : \mathrm{mor}_{\mathcal{C}}(Y, B) \to \mathrm{mor}_{\mathcal{C}}(X, B)$ for the function $f^* : g \mapsto g \circ f$. Then the rules*

  $$F(X) = \mathrm{mor}_{\mathcal{C}}(X, B) \quad \text{and} \quad F(f) = f^* : F(Y) \to F(X)$$

  define a contravariant functor from $\mathcal{C}$ to Sets.
- (b) *For $f : X \to Y$, write $f_* : \mathrm{mor}_{\mathcal{C}}(A, X) \to \mathrm{mor}_{\mathcal{C}}(A, X)$ for the function $f_* : g \mapsto f \circ g$. Then the rules*

  $$G(X) = \mathrm{mor}_{\mathcal{C}}(A, X) \quad \text{and} \quad G(f) = f_* : G(X) \to G(Y)$$

  define a covariant functor from $\mathcal{C}$ to Sets.

A functor that is constructed in either of these ways is called a **represented functor** – that is, the functor F is represented by the object B, and the functor G is represented by the object A.

PROBLEM 1.21 Prove Proposition 4.

HINT Simply generalize your work from Exercise 1.20.

PROBLEM 1.22 Let $f : A \to B$ be a morphism in the category $\mathcal{C}$.

- (a) Suppose the induced map $f_* : \mathrm{mor}_{\mathcal{C}}(X, A) \to \mathrm{mor}_{\mathcal{C}}(X, B)$ is a bijection for every X. Show that f is an equivalence.
- (b) Suppose the induced map $f^* : \mathrm{mor}_{\mathcal{C}}(B, X) \to \mathrm{mor}_{\mathcal{C}}(A, X)$ is a bijection for every X. Show that f is an equivalence.

HINT Try plugging in $X = A$ and $X = B$.

1.4 Natural Transformations

Category theory was invented by Saunders Mac Lane and Samuel Eilenberg in the early 1940's, largely motivated by the desire to be precise about what is meant by (or should be meant by) a 'natural construction.' For many years before then, mathematicians had used the *intuitive* notion of a natural construction to mean that the construction is done in exactly the same way for all spaces, groups, or whatever. For example, for any vector space V, you can construct the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{R})$; since this is done in the same way for every vector space, it is 'naturally defined' and so it will 'of course' (ha!) convert commutative diagrams to other commutative diagrams. This idea is formalized in the idea of a functor. Now how do we relate two different 'natural' constructions to one another?

Definition 5 Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to D$ be two covariant [5] functors. A **natural transformation** $\Phi : F \to G$ is a rule that associates to each $X \in \mathrm{ob}(\mathcal{C})$ a morphism

$$\Phi_X : F(X) \to G(X)$$

with the property that for any morphism $f : X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\Phi_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\Phi_Y"] & G(Y) \end{tikzcd}$$

is commutative.

It is easy to find examples of natural transformations between represented functors.

Problem 1.23 Let $\phi : A \to B$ be a morphism in $\mathcal{C}$.

(a) Define two functors $\mathcal{C} \to$ Sets by the rules $F(X) = \mathrm{mor}_{\mathcal{C}}(X, A)$ and $G(X) = \mathrm{mor}_{\mathcal{C}}(X, B)$. Show that $\Phi_X = \phi_* : F \to G$ is a natural transformation.

(b) Define two functors $\mathcal{C} \to$ Sets by the rules $H(X) = \mathrm{mor}_{\mathcal{C}}(A, X)$ and $I(X) = \mathrm{mor}_{\mathcal{C}}(B, X)$. Show that $\Phi_X = \phi^* : I \to H$ is a natural transformation.

Exercise 1.24 This problem refers to the functors $F(V) = \hom(\mathbb{R}, V)$ and $G(V) = \hom(V, \mathbb{R})$ of Exercise 1.20.

(a) Show that for every vector space V, $V \cong F(V)$. Define a natural isomorphism $\Phi : F \to \mathrm{Id}$.

(b) Show that for every finite dimensional vector space V, $V \cong G(V)$. Show that there is no **natural** isomorphism $\Theta : G \to \mathrm{Id}$, even if you restrict your attention just to finite dimensional vector spaces.

In fact, the converse of Problem 1.23 is true – this is known as the *Yoneda Lemma.*

Proposition 6 *Let $A, B \in \mathcal{C}$.*

(a) Define $H, I : \mathcal{C} \to$ Sets by the rules $H(X) = \mathrm{mor}_{\mathcal{C}}(A, X)$ and $I(X) = \mathrm{mor}_{\mathcal{C}}(B, X)$. Then there is a bijection

$$\{\text{Natural Transformations } I \to H\} \quad \to \quad \mathrm{mor}_{\mathcal{C}}(A, B).$$

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[5] They could also both be contravariant. I'll leave the formulation to you.

(b) *Define* $F, G : \mathcal{C} \to \text{Sets}$ *by the rules* $F(X) = \text{mor}_{\mathcal{C}}(X, A)$ *and* $G(X) = \text{mor}_{\mathcal{C}}(X, B)$. *Then there is a bijection*

$$\{\text{Natural Transformations } F \to G\} \quad \to \quad \text{mor}_{\mathcal{C}}(A, B).$$

Your next problem is to prove the Yoneda Lemma.

PROBLEM 1.25 Let $F, G, H, I : \mathcal{C} \to \text{Sets}$ be the functors defined in Problem 1.23.

(a) If $\Phi : F \to G$ is a natural transformation, show that there is a unique map $\phi : A \to B$ such that $\Phi_X = \phi_*$ for every $X \in \mathcal{C}$.

(b) If $\Phi : I \to H$ is a natural transformation, show that there is a unique map $\phi : A \to B$ such that $\Phi_X = \phi^*$ for every $X \in \mathcal{C}$.

Notice that your proof of (b) is formally very similar to your proof of (a). Can you be precise about how the two proofs are related?

HINT In both cases, the domain of ϕ is A, and the target is B.

PROBLEM 1.26 Let $A, B \in \mathcal{C}$, and use them to define functors

$$F(X) = \text{mor}_{\mathcal{C}}(X, A) \quad \text{and} \quad G(X) = \text{mor}_{\mathcal{C}}(X, B).$$

(a) Suppose there is a natural isomorphism $\Phi : F \to G$. Show that $A \cong B$.

(b) Show that (a) is false without the word 'natural' – that is, make up an example where $F(X) \cong G(X)$ for all X, but where, nonetheless, $A \ncong B$.

HINT Your category must have at least two objects; can it have exactly two objects?

(c) Prove that A and B are isomorphic if the (contravariant) functors $\text{mor}_{\mathcal{C}}(A, ?\,)$ and $\text{mor}_{\mathcal{C}}(B, ?\,)$ are naturally equivalent.

Natural Transformations in Dumber Categories. Before ending this section, we mention a wrinkle on the definition of a natural transformation. The intuitive idea of a natural transformation is that it is some construction which is done 'in the same way for all objects.' With this definition, consider the category Rings of all rings and their homomorphisms. For each ring R, we can define

$$\phi_R : R \to R \qquad \text{by the rule} \qquad x \mapsto x^2.$$

This rule clearly fits in to the *intuitive* idea of a natural transformation $\Phi : I \to I$, where $I : \text{Rings} \to \text{Rings}$ is the identity functor. But it is not a natural transformation, *because* ϕ_R *is not a ring homomorphism.*

To make ϕ a natural transformation, we need to move to a category in which the maps are not required to be ring homomorphisms. One solution would be to let Rings_0 be the category whose objects are rings and whose morphisms are maps of sets. Then there is a forgetful functor $F : \mathrm{Rings} \to \mathrm{Rings}_0$, and ϕ is a natural transformation from F to itself.

Thus we will sometimes find it useful allow our natural transformations $\Phi : F \to G$ (where $F, G : \mathcal{C} \to \mathcal{D}$) to give maps $\phi_X : F(X) \to G(X)$ that are not maps in $\mathcal{D}$ but maps in some larger category that contains $\mathcal{D}$.

1.5 Duality

In studying categories, you should keep your eyes open for instances of duality. The dual of a category-theoretical expression is the result of reversing all the arrows, changing each reference to a domain to refer to the target (and vice versa), and reversing the order of composition.

Exercise 1.27

(a) The notation $f : X \to Y$ is shorthand for the sentence: 'f is a morphism with domain X and target Y.' What is the dual of this statement?

(b) Find instances of duality in the previous sections.

For example, consider **lifting problem**: you are given maps $f : A \to Y$ and $p : X \to Y$, and you would like to find a map $\lambda : A \to X$ such that $p \circ \lambda = f$. This problem is neatly expressed in the diagram

$$\begin{tikzcd}
 & X \arrow[d, "p"] \\
A \arrow[ur, dotted, bend left, "\lambda"] \arrow[r, "f"] & Y
\end{tikzcd}$$

(when expressing problems in this way, the map you hope to find is usually dotted or dashed). The dual problem is expressed by the diagram

$$\begin{tikzcd}
 & V \arrow[dl, dotted, bend right, "\epsilon"'] \\
B & U \arrow[l, "g"'] \arrow[u, "q"']
\end{tikzcd}$$

This is known as an **extension problem**, because you hope to extend the map g to the 'larger' thing V.[6]

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[6] Norman Steenrod, one of the architects of modern Algebraic Topology, used the ex-

Problem 1.28 Verify that the dual of each rule for a category is also a rule for a category. Likewise for functors and natural transformations.

Because of Problem 1.28, the dual of a valid proof involving categories, functors and natural transformations is also a valid proof. Thus, the dual of every theorem of pure category theory is automatically also a theorem.

Domain- and Target-Type Objects. It often happens that an object of a category is defined in terms of certain category-theoretical properties. These properties usually give special information about the maps *out of* the object, or else they give special information about the morphisms *into* that object. In the first case, we call the construction a construction of **domain type**; in the second case it is a construction of **target type**. We will sometimes refer to the results of these constructions as being objects of 'domain type' or of 'target type.' The distinction between 'domain type' and 'target type' objects or constructions is important to observe. The dual of a domain type construction is a target type construction and vice versa.

1.6 Products and Sums

Let $X, Y \in \mathcal{C}$. The **product** of X and Y is an object P *together with* two morphisms $\mathrm{pr}_X : P \to X$ and $\mathrm{pr}_Y : P \to Y$ (called **projections**) with the following **universal property**: if $f : Z \to X$ and $g : Z \to Y$ are any two morphisms, then there is a *unique* morphism $t : Z \to P$ so that $\mathrm{pr}_X \circ t = f$ and $\mathrm{pr}_Y \circ t = g$. This definition can be expressed diagramatically as follows:

$$\begin{tikzcd}
 & Z \arrow[dl, bend right, "f"'] \arrow[dr, bend left, "g"] \arrow[d, dotted, "\exists!\ t"] & \\
X & P \arrow[l, "\mathrm{pr}_X"] \arrow[r, "\mathrm{pr}_Y"'] & Y
\end{tikzcd}$$

There is no guarantee that two given objects in a category $\mathcal{C}$ actually have a product, or that there will only be one product.

Problem 1.29 Suppose $X, Y \in \mathcal{C}$, and the objects P and Q are both products for X and Y. Show that $P \cong Q$.

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tension and lifting problems to frame the entire subject. It can be argued that a great deal of mathematics is about lifting and extension problems. Exercise: *Show how the problem: 'decide whether $f : X \to Y$ is a homeomorphism' can be written in terms of extension and/or lifting problems.*

Since any two products are equivalent, we often just choose one of them, and denote it by $X \times Y$.

In this diagram, the object being defined is P, and the dotted arrow indicates a hypothetical arrow. This is also common practice in defining concepts using diagrams. Since the object P is defined in terms of its properties as a *target* (i.e., it is the target of the hypothetical arrow), the product is a target-type construction.

EXERCISE 1.30 Let $X, Y \in \mathcal{C}$, and suppose $X \times Y$ exists. Explicitly define a bijection

$$\operatorname{mor}_{\mathcal{C}}(Z, X \times Y) \xrightarrow{\cong} \operatorname{mor}_{\mathcal{C}}(Z, X) \times \operatorname{mor}_{\mathcal{C}}(Z, Y).$$

Because of this, we will generally write maps $F : Z \to X \times Y$ in the form (f, g), where $f = \mathrm{pr}_1 \circ F$ and $g = \mathrm{pr}_2 \circ F$. One particularly important map is the **diagonal map**

$$\Delta : X \to X \times X \quad \text{defined by} \quad \Delta = (\mathrm{id}_X, \mathrm{id}_X).$$

More generally, if J is some set, then we can define the J-fold product of X with itself, and (if it exists) there is a diagonal map $\Delta_J : X \to \prod_{j \in J} X$, which is the unique map such that $\mathrm{pr}_j \circ \Delta_J = \mathrm{id}_X$ for each $j \in J$.

EXERCISE 1.31

(a) Show that if one of the products $X \times (Y \times Z)$ and $(X \times Y) \times Z$ exists in $\mathcal{C}$, then so does the other, and they are isomorphic.

(b) Let $f : A \to X$ and $g : B \to Y$, and suppose that the products $A \times B$ and $X \times Y$ can be formed in $\mathcal{C}$. Give an explicit definition for the product map
$$f \times g : A \times B \to X \times Y.$$

Suppose that $\mathcal{C}$ is a category with the property that *every* pair of objects $X, Y \in \mathcal{C}$ has a product. Then by choosing one product $X \times Y$ for each pair, we see that Exercise 1.31(b) implies that the rules $(X, Y) \mapsto X \times Y$ and $(f, g) \mapsto f \times g$ define a functor of two variables

$$\times : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

Expand on this!

Remark 7 Some authors use the notation $f \times g$ to denote the map $Z \to X \times Y$ with components f and g. But this is wrong!

EXERCISE 1.32 Formulate precise definitions of the product $\mathcal{C} \times \mathcal{D}$ of two categories and of a functor of two variables.

Problem 1.33 Let X and Y be objects in a category $\mathcal{C}$.

(a) Show that, for any map $f : X \to Y$, the diagram

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[d, "f"'] & X \times X \arrow[d, "f \times f"] \\
Y \arrow[r, "\Delta"] & Y \times Y
\end{tikzcd}$$

commutes.

(b) Show that the diagram

$$\begin{tikzcd}
X \times Y \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & (X \times Y) \times (X \times Y) \arrow[d, "(\mathrm{pr}_1, \mathrm{pr}_2)"] \\
& X \times Y
\end{tikzcd}$$

is commutative.

It is tempting to prove these by chasing elements around, which is fine if your objects are sets. But you should prove these diagrams commute only using the category theoretical definitions of the maps involved.

Exercise 1.34 Explain how to view the diagonal map as a natural transformation.

Let's look at some specific examples of products.

Exercise 1.35

(a) Show that the product of two sets X and Y is simply the ordinary cartesian product

$$X \times Y = \{(x, y) \,|\, x \in X, y \in Y\}.$$

(b) What is the product of two abelian groups G and H?

Let's look at the dual concept.

Problem 1.36

(a) Formulate the (dual) definition of a **coproduct**, which is denoted $X \sqcup Y$. Coproducts are also known as (categorical) **sums**.

(b) Prove that if $X, Y \in \mathrm{ob}(\mathcal{C})$, then

$$\mathrm{mor}(X \sqcup Y, B) \cong \mathrm{mor}(X, B) \times \mathrm{mor}(Y, B).$$

Write down the isomorphism explicitly. Thus we can (and will) describe maps $F : X \sqcup Y \to B$ with the notation (f, g), where $f : X \to B$ and $g : Y \to B$.

Hint This is formally dual to Exercise 1.30, so you should be able to prove this by simply inverting all the arrows in your previous proof.

(c) Write down the definition of $f \sqcup g : A \sqcup B \to X \sqcup Y$.

(d) Explain how to view $\sqcup$ as a functor.

(e) The dual of the diagonal map is called the **folding map**, and we will denote it by the symbol ∇. Define it explicitly in category theoretic language, and explain how to view it as a natural transformation.

The coproduct, being dual to the product, is a domain-type construction.

EXERCISE 1.37

(a) Show that the sum of two sets X and Y is simply the disjoint union of X and Y. Conclude that $X \sqcup Y$ and $X \times Y$ are not generally equivalent.

(b) What is the folding map in the case $X = \{a, b, c\}$?

(c) What is the sum of abelian groups G and H? Construct a nice map $w : G \sqcup H \to G \times H$; what can you say about it?

EXERCISE 1.38

(a) Give number-theoretical intepretations of products and sums in the category of positive integers $1, 2, 3, \ldots$ with morphisms corresponding to divisibility.

(b) Repeat (a) with the category of real numbers with morphisms corresponding to inequalities $x \leq y$.

(c) Is there a category structure on the set $\mathbb{N}$ so that the categorical product is the same as the numerical product?

1.7 Initial and Terminal Objects

An object $\tau \in \mathcal{C}$ is called a **terminal object** if the set $\mathrm{mor}_{\mathcal{C}}(X, \tau)$ has exactly one element, no matter what $X \in \mathcal{C}$ we plug in.[7] Dually, an object $\iota \in \mathcal{C}$ is called an **initial object** if the set $\mathrm{mor}_{\mathcal{C}}(\iota, Y)$ has exactly one element, no matter what $Y \in \mathcal{C}$ we plug in.

EXERCISE 1.39 Find initial and terminal objects in the following contexts.

(a) The category of sets and functions.

(b) The category of topological spaces and continuous functions.

(c) The category of groups and homomorphisms.

PROBLEM 1.40

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[7] So a terminal object is a target-type concept.

(a) Suppose $\mathcal{C}$ is a category with a terminal object τ, and let $X, Y \in \mathcal{C}$, and suppose that a product P for X and Y exists in $\mathcal{C}$. Show that P solves the problem expressed in the diagram

$$\begin{tikzcd}
Z \arrow[drr, bend left, "f"] \arrow[dr, dotted, "\exists!\, t"] \arrow[ddr, bend right, "g"'] & & \\
& P \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] & X \arrow[d] \\
& Y \arrow[r] & \tau.
\end{tikzcd}$$

(There is never any need to label a map into a terminal object!)

(b) Formulate and prove the dual to part (a).

A **pointed category** is a category $\mathcal{C}$ in which there is an object, generally[8] denoted $*$, which is simultaneously initial and terminal. If $X, Y \in \mathcal{C}$, where $\mathcal{C}$ is a pointed category, then there is a unique morphism of the form $X \to * \to Y$; it is called the **trivial morphism**, and it will be uniformly denoted $*$.

In a pointed category, the sum of X and Y is sometimes denoted $X \vee Y$, and referred to as the **wedge** sum of X and Y.

An important example of a pointed category is the category $\mathcal{S}_*$ of **pointed sets**. An object of $\mathcal{S}_*$ is a set X with a particular point x_0 chosen and identified; it is referred to as the **basepoint** of X. A morphism from (X, x_0) to (Y, y_0) is a function $f : X \to Y$ with the additional property that $f(x_0) = y_0$. In practice, we do not give individual names to the basepoints, but just call them all $*$.

Exercise 1.41 Verify that $\mathcal{S}_*$ is a pointed category. What is a sum in $\mathcal{S}_*$? What is a product?

Problem 1.42 Let $\mathcal{C}$ be pointed category in which products and coproducts exist for all pairs of objects.

(a) Give categorical definitions for the X and Y 'axis' maps $\mathrm{in}_X : X \to X \times Y$ and $\mathrm{in}_Y : Y \to X \times Y$.

(b) In a pointed category, there is a particularly nice morphism $w : X_1 \vee X_2 \to X_1 \times X_2$. Define it in terms of category theory, and check that the diagram

$$\begin{tikzcd}
X_1 \vee X_2 \arrow[r, "f \vee g"] \arrow[d, "w"'] & Y_1 \vee Y_2 \arrow[d, "w"] \\
X_1 \times X_2 \arrow[r, "f \times g"] & Y_2 \times Y_2
\end{tikzcd}$$

---

[8] Though in algebraic contexts, it is often 0 or $\{1\}$ or something even more substantial.

is commutative for any $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$. Why is it necessary for the category to be pointed before you can define w?

(c) State (and prove?) the dual statements.

Exercise 1.43 In what sense do the maps w constitute a natural transformation?

Problem 1.44 Show that for any $f : X \to Y$ in a pointed category and $* : W \to X$, then $f \circ * = * : W \to Y$. Also show that if $* : Y \to Z$, then $* \circ f = * : X \to Z$. Conclude that the functors $\mathrm{mor}_{\mathcal{C}}(\,?\,, Y)$ and $\mathrm{mor}_{\mathcal{C}}(A, \,?\,)$ take their values in the category of pointed sets and pointed maps.

Exercise 1.45

(a) Show that the trivial group $\{1\}$ is simultaneously initial and terminal in the category of groups and homomorphisms. Show that the vector space 0 is simultaneously initial and terminal in the category of vector spaces (over $\mathbb{R}$, if you like) and linear transformations.

(b) Show that in the category of abelian groups and homomorphisms, the map $w : G \vee G \to G \times H$ is an isomorphism for any G and H. Also show that the analogous statement is true in the category of vector spaces and linear transformations.

Since sums are domain-type constructions and products are target-type constructions, the maps from a sum to a target should be fairly easy to understand.

Problem 1.46

(a) Show that, in any category $\mathcal{C}$, there is a natural bijection between the morphism set $\mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Y_1 \times Y_2)$ and the set M of all matrices

$$\begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

with $f_{ij} \in \mathrm{mor}_{\mathcal{C}}(X_j, Y_i)$.

(b) Now suppose that $\mathcal{C}$ is a pointed category in which the canonical maps $w : X \vee Y \to X \times Y$ is an isomorphism for each pair of objects $X, Y \in \mathcal{C}$. Show that composition

$$\begin{tikzcd}
\mathrm{mor}_{\mathcal{C}}(Y_1 \sqcup Y_2, Z_1 \times Z_2) \times \mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Y_1 \times Y_2) \arrow[r] \arrow[d, "\cong"'] & \mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Z_1 \times Z_2) \\
\mathrm{mor}_{\mathcal{C}}(Y_1 \times Y_2, Z_1 \times Z_2) \times \mathrm{mor}_{\mathcal{C}}(X_1 \sqcup X_2, Y_1 \times Y_2) \arrow[ur] &
\end{tikzcd}$$

corresponds to matrix multiplication in M.[9]

(c) Show that linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ are in one-to-one correspondence with 2×2 matrices with real entries.

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[9] What exactly do I mean by 'matrix multiplication'?

1.8 Group and Cogroup Objects

A group is a set G with a multiplication (which can be thought of as the map $\mu : G \times G \to G$ given by $(x, y) \mapsto x \cdot y$) and, for each element, an inverse (which can be thought of as the map $\nu : G \to G$ given by $x \mapsto s^{-1}$), which are required to satisfy various properties. The main observation of this section is that all of these properties can be formulated abstractly in terms of diagrams.

Definition 8 Let $\mathcal{C}$ be a pointed category, and $G \in \mathcal{C}$. Then G is a **group object** if there are maps

$$\begin{array}{ll} \mu : G \times G \to G & \text{(multiplication), and} \\ \nu : G \to G & \text{(inverse)} \end{array}$$

which satisfy the following properties:

1. (Identity) The following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "(*{,}\mathrm{id}_G)"] \arrow[dr, bend right=15, "\mathrm{id}_G"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "(\mathrm{id}_G{,}*)"'] \arrow[dl, bend left=15, "\mathrm{id}_G"] \\
& G. &
\end{tikzcd}$$

2. (Inverse) The following diagram commutes:

$$\begin{tikzcd}
G \arrow[r, "(\nu{,}\mathrm{id}_G)"] \arrow[dr, bend right=15, "*"'] & G \times G \arrow[d, "\mu"] & G \arrow[l, "(\mathrm{id}_G{,}\nu)"'] \arrow[dl, bend left=15, "*"] \\
& G &
\end{tikzcd}$$

3. (Associativity) The following diagram commutes:

$$\begin{tikzcd}
G \times G \times G \arrow[r, "\mu \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mu"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G.
\end{tikzcd}$$

It is sometimes useful to study objects with are not quite group objects. For example, if we drop the inverse conditions, we obtain a **monoid object**. Further, we can forgo associativity; objects with this limited structure are called ???. look this up

PROBLEM 1.47 Let $\mathcal{C}$ be a category such that every pair $X, Y \in \mathcal{C}$ has both a product and a coproduct. Show that $M \in \mathcal{C}$ is a ??? if and only if in every diagram

$$\begin{tikzcd}
X \vee Y \arrow[r, "f"] \arrow[d] & M \\
X \times Y \arrow[ur, dotted] &
\end{tikzcd}$$

the dotted arrow can be filled in to make the triangle commute.

EXERCISE 1.48 Are group objects domain-type or target-type gadgets? Suppose G is a group object in $\mathcal{C}$. What conditions must you impose on a functor $F : \mathcal{C} \to \mathcal{D}$ in order to conclude that $F(G) \in \mathcal{D}$ is also a group object?

Let's check that these things are correctly named.

EXERCISE 1.49

(a) Check that, in the category of pointed sets, a group object is just an ordinary group.

(b) Let $\mathcal{G}$ be the category of groups and homomorphism. Show that a group $G \in \mathcal{G}$ is a group object if and only if G is abelian.

(c) Write $GL_n(\mathbb{R})$ to denote the set of all $n \times n$ invertible matrices. It is a subset of $\mathbb{R}^{n^2}$, so we can give it the subspace topology. Show that matrix multiplication makes $GL_n(\mathbb{R})$ into a group object in the category of pointed topological spaces.

EXERCISE 1.50 You know that in the category of pointed sets and their maps, the inverse map ν for a group object G is uniquely determined by its multiplication μ. Prove that this is true for group objects in any category.

The reason group objects are so important is that they provide a group structure to morphism sets.

PROBLEM 1.51 Let $\mathcal{C}$ be any pointed category and let $G \in \mathcal{C}$ be a group object.

(a) Show that the composite map M in the diagram

$$\begin{tikzcd}
\mathrm{mor}_{\mathcal{C}}(X, G) \times \mathrm{mor}_{\mathcal{C}}(X, G) \arrow[r, "M"] \arrow[d, "\cong"'] & \mathrm{mor}_{\mathcal{C}}(X, G) \\
\mathrm{mor}_{\mathcal{C}}(X, G \times G) \arrow[ur, "\mu_*"'] &
\end{tikzcd}$$

makes $\mathrm{mor}_{\mathcal{C}}(X, G)$ into a group object in the category of pointed sets (i.e., $\mathrm{mor}_{\mathcal{C}}(X, G)$ is a group with multiplication M).

(b) Draw a diagram that shows all the maps involved in the definition of the product of $\alpha, \beta \in \mathrm{mor}_{\mathcal{C}}(X, G)$, and how they fit together. (In other words, write down explicitly what $\alpha \cdot \beta$ is.)

(c) Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Show that $f^* : \mathrm{mor}_{\mathcal{C}}(Y, G) \to \mathrm{mor}_{\mathcal{C}}(X, G)$ is a group homomorphism.

Hint Use Problem 1.33 and part (b).

Let's think about what you have just proved. We have two functors:

$$F : \mathcal{C} \to \mathcal{S}_* \quad \text{and} \quad \text{forget} : \mathcal{G} \to \mathcal{S}_*.$$

which can be arranged like so:

$$\begin{tikzcd} & \mathcal{G} \arrow[d, "\text{forget}"] \\ \mathcal{C} \arrow[r, "F"'] \arrow[ur, dotted, bend left] & \mathcal{S}_*. \end{tikzcd}$$

You have proved that the dotted arrow can be filled in, and thereby proved the following theorem.

Theorem 9 *If G is a group object in a pointed category $\mathcal{C}$, then the contravariant functor $F(X) = \mathrm{mor}_{\mathcal{C}}(X, G)$ factors through the forgetful functor from the category $\mathcal{G}$ of groups and homomorphisms to the category of pointed sets $\mathcal{S}_*$.*

We usually use the same symbol, F, for the dotted functor $\mathcal{C} \to \mathcal{G}$. It is sometimes said in this situation that F 'takes its values in the category $\mathcal{G}$.' This phrasing, thought not entirely accurate, makes sense because $\mathcal{G}$ can be considered to be a subcategory of $\mathcal{S}_*$ (via the forgetful functor).

Exercise 1.52 Let G be a group object in a category $\mathcal{C}$. Work out the product $\mathrm{pr}_1 \cdot \mathrm{pr}_2 \in \mathrm{mor}_{\mathcal{C}}(G \times G, G)$. You should be able to express it as a specific map you already know.

Now let's dualize.

Problem 1.53 Write down the definition of a cogroup object. What is a cogroup object in the category of groups and homomorphisms? What about abelian groups and homomorphisms? What is a cogroup object in the category of pointed sets? What if you replace 'cogroup' with 'comonoid'?

By dualizing our discussion of group objects, we can immediately derive the following result.

Theorem 10 *If C is a cogroup object in a pointed category $\mathcal{C}$, then the covariant functor $G(Y) = \mathrm{mor}_{\mathcal{C}}(C, Y)$ takes its values in the category of groups and homomorphisms.*

Problem 1.54 Prove Theorem 10.

Suppose C is a cogroup object and G is a group object. Then the set $\mathrm{mor}(X, Y)$ has *two* ways to multiply. More precisely, $\mathrm{mor}_{\mathcal{C}}(C, G)$ is a group because C is a cogroup object – we'll write $\alpha\spadesuit\beta$ for this product; and $\mathrm{mor}_{\mathcal{C}}(C, G)$ is a group because G is a group object – we'll write $\alpha\heartsuit\beta$ for this product.

Problem 1.55 Show that, with the setup above, the products $\spadesuit$ and $\heartsuit$ are the same. That is, show that for any $f, g \in \mathrm{mor}_{\mathcal{C}}(C, G)$, $f\spadesuit g = f\heartsuit g$.

Hint Write down the compositions which define $f\spadesuit g$ and $f\heartsuit g$ in a single commutative diagram. Use Problem 1.42.

In view of Problem 1.55, we will never again use suits to denote products, and will be content to write $f \cdot g$, or simply fg for these products.

1.9 Homomorphisms

As you know from your study of algebra, when you are studying groups, you inevitably find yourself studying homomorphisms. Our goal in this section is to establish definitions for homomorphisms of group and cogroup objects, and to prove some simple but important facts about them.

Definition 11 Let G and H be group objects in a pointed category $\mathcal{C}$. A map $f : G \to H$ is a **homomorphism** if the diagram

$$\begin{tikzcd}
G \times G \arrow[r, "\mu_G"] \arrow[d, "f\times f"'] & G \arrow[d, "f"] \\
H \times H \arrow[r, "\mu_H"] & H
\end{tikzcd}$$

commutes.[10]

Exercise 1.56 Show that when $\mathcal{C}$ is the category of pointed sets, a homomorphism of group objects is just the same as a homomorphism of groups.

Exercise 1.57 Show that if $f : G \to H$ is a homomorphism of group objects in $\mathcal{C}$, then f preserves inverses, in the sense that the diagram

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\nu_G"'] & H \arrow[d, "\nu_H"] \\
G \arrow[r, "f"] & H
\end{tikzcd}$$

---

[10] This is actually a perfectly good definition for a monoid homomorphism.

is commutative.

If G and H are group objects, and $f : G \to H$ is some map, then we automatically get an induced map

$$f_* : \mathrm{mor}_{\mathcal{C}}(X, G) \to \mathrm{mor}_{\mathcal{C}}(X, H).$$

From what we know already, the sets $\mathrm{mor}_{\mathcal{C}}(X, G)$ and $\mathrm{mor}_{\mathcal{C}}(X, H)$ are groups; but what can we say about the map f_*? In general, there is nothing we can say, but if f is a homomorphism of group objects, then f_* is a group homomorphism.

Theorem 12 *If $f : G \to H$ is a homomorphism of group objects in the pointed category $\mathcal{C}$, then the induced map*

$$f_* : \mathrm{mor}_{\mathcal{C}}(X, G) \to \mathrm{mor}_{\mathcal{C}}(X, H).$$

is a homomorphism of groups for every $X \in \mathcal{C}$.

PROBLEM 1.58 Prove Theorem 12. Is the converse true?

As usual, it is up to you to formulate the duals.

PROBLEM 1.59 Define homomorphisms of cogroup objects, and prove that they induce group homomorphisms on mapping sets.

1.10 Abelian Groups and Cogroups

Since abelian groups are especially easy to work with, we establish the notion of commutative groups and cogroups.

Abelian Objects. In any category, we can define a **twist map** for coproducts $T : X \sqcup Y \to Y \sqcup X$, which 'switches the terms.'

PROBLEM 1.60 Write down a categorical description of T. Also define the twist map $T : X \times Y \to Y \times X$ for products. Show that both twist maps are equivalences.

Definition 13 A cogroup object C is **cocommutative**, or simply **commutative**, if the diagram

$$\begin{tikzcd} & C \arrow[dl, "\phi"'] \arrow[dr, "\phi"] & \\ C \vee C \arrow[rr, "T"'] & & C \vee C \end{tikzcd}$$

is commutative.

Problem 1.61 Show that C is a cocommutative cogroup if and only if $\text{mor}_{\mathcal{C}}(C, Y)$ is an abelian group for every Y.

Problem 1.62 Dualize this discussion: define a commutative group object, and verify that $\text{mor}_{\mathcal{C}}(X, G)$ is abelian if and only if G is such an object.

Products of Groups. As you know, the set-theoretical product of two groups can be made into a group using coordinatewise multiplication. The same can be done with group objects. Let G and H be group objects in the pointed category $\mathcal{C}$, and denote their multiplications by μ_G and μ_H. Then $G \times H$ can be made into a group object in $\mathcal{C}$ using the multiplication given by

$$\begin{tikzcd}
(G \times H) \times (G \times H) \arrow[r, "\text{id} \times T \times \text{id}"] \arrow[dr, bend right=10, "\mu_{G \times H}"'] & (G \times G) \times (H \times H) \arrow[d, "\mu_G \times \mu_H"] \\
& G \times H
\end{tikzcd}$$

and inverse $\nu_{G \times H} = \nu_G \times \nu_H$.

Problem 1.63

(a) Show that the maps $\mu_{G \times H}$ and $\nu_{G \times H}$ make $G \times H$ into a group object in $\mathcal{C}$.

(b) Show that the inclusions $G \to G \times H$ and $H \to G \times H$ are homomorphisms.

(c) Show that the projections $G \times H \to G$ and $G \times H \to H$ are homomorphisms.

The definition of the product in $G \times G$ involves the map $T : G \times G \to G \times G$, which was introduced in order to define commutative groups.

Problem 1.64 Show that a group object is commutative if and only if the multiplication $\mu : G \times G \to G$ is a homomorphism. What is the dual statement?

1.11 Adjoint Functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ be two (covariant) functors. Then for $X \in \mathcal{D}$ and $Y \in \mathcal{C}$, we can form

$$\text{mor}_{\mathcal{C}}(L(X), Y) \quad \text{and} \quad \text{mor}_{\mathcal{D}}(X, R(Y)),$$

which defines functors

$$M, N : \mathcal{D} \times \mathcal{C} \to \text{Sets}$$

which are contravariant in the first coordinate and covariant in the second. In general, of course, there need not be any relationship between these two sets. But in many important cases, there is a natural *isomorphism* $\Phi : M \to N$; that is, for each $X \in \mathcal{D}$ and $Y \in \mathcal{C}$, there is an equivalence

$$\Phi_{X,Y} : \mathrm{mor}_{\mathcal{C}}(L(X), Y) \to \mathrm{mor}_{\mathcal{D}}(X, R(Y)),$$

and these equivalences respect maps between domains and maps between targets. When this occurs, we say that the functors L and R are **adjoint** to one another. More precisely, L is the **left adjoint** and R is the corresponding **right adjoint**. Whenever we introduce an adjoint pair of functors as L and R, we are making the tacit assertion that L is the left adjoint and R is the right adjoint.

We will use the notation $\hat{\alpha} = \Phi_{X,Y}(\alpha)$. Thus, if $\alpha : LX \to Y$, then $\hat{\alpha} : X \to RY$.

EXERCISE 1.65 Let $f : X \to X'$ and $g : Y \to Y'$. Write down the diagrams which must commute in order for Φ to be a natural transformation.

PROBLEM 1.66 Let L and R be adjoint.

(a) Show that if one of the squares

$$\begin{tikzcd} LA \arrow[r, "\alpha"] \arrow[d, "Lf"'] & X \arrow[d, "g"] \\ LB \arrow[r, "\beta"] & Y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "\hat{\alpha}"] \arrow[d, "f"'] & RX \arrow[d, "Rg"] \\ B \arrow[r, "\hat{\beta}"] & RY \end{tikzcd}$$

commutes, then so does the other one.

(b) Let F, G be functors. Show that $\Phi : LF \to G$ is a natural transformation if and only if $\hat{\Phi} : F \to RG$ is a natural transformation.

EXERCISE 1.67 If X is a set, then we can form the **free abelian group** $F(X) = \bigoplus_{x \in X} \mathbb{Z}$. On the other hand, if G is an abelian group, then we can forget the group structure of G, and just remember the underlying set $S(G)$.

(a) Show that the functors $F : \text{Sets} \to \text{Groups}$ and $S : \text{Groups} \to \text{Sets}$ are adjoint to one another. Which is the left adjoint and which is the right adjoint?

(b) Use the same scheme to express other 'free objects' that you know about in terms of adjoints.

Let's start with functors $R : \mathcal{C} \to \mathcal{D}$ and $L : \mathcal{D} \to \mathcal{C}$ which are adjoint. Taking $X = L(Y)$, we have a natural isomorphism

$$\Phi : \mathrm{mor}_{\mathcal{D}}(L(Y), L(Y)) \to \mathrm{mor}_{\mathcal{C}}(Y, RL(Y)).$$

Applying this to the identity $\mathrm{id}_{L(Y)}$ gives us a map $\sigma : Y \to RL(Y)$.

PROBLEM 1.68 Show that there is a commutative diagram

$$\begin{tikzcd}
\mathrm{mor}_{\mathcal{C}}(X, Y) \arrow[r, "L"] \arrow[d, equal] & \mathrm{mor}_{\mathcal{C}}(L(X), L(Y)) \arrow[d, "\Phi", "\cong"'] \\
\mathrm{mor}_{\mathcal{C}}(X, Y) \arrow[r, "\sigma_*"] & \mathrm{mor}_{\mathcal{C}}(X, RL(Y)).
\end{tikzcd}$$

Thus, the effect of L on morphisms can be identified with a map defined by composition of morphisms.

EXERCISE 1.69 Problem 1.68 shows that the maps L and σ_* are **equivalent maps**. In Section 1.2 we defined what equivalence means in categorical terms – what category are we working in here?

Dually, if we take $Y = R(X)$, then we have an isomorphism

$$\Phi : \mathrm{mor}_{\mathcal{C}}(R(X), R(X)) \to \mathrm{mor}_{\mathcal{D}}(LR(X), X),$$

and we define $\lambda : LR(X) \to X$ to be the image of $\mathrm{id}_{R(X)}$.

PROBLEM 1.70 Dualize the previous problem.

1.12 Projects

These should be nontrivial open ended projects related to the material in the chapter.

1. Write down the basic setup of Galois theory in category theoretical language. Carefully define all categories and functors involved.

2. Why do not require that we have a *set* of objects in a category? Why is it okay to require a set of morphisms? Why is it desirable to require a set of morphisms? Is there a category whose objects are categories and whose morphisms are functors?

3. Refer back to our discussion of forgetful functors. This involved the phrase ‘dumbed down,’ which is admittedly not very precise. Figure out a way to make it precise, and so give an formal definition of ‘forgetful functor.’

4. Algebraic theories....

5. We have talked about adjoint functors, and you know about the adjoint of a matrix. Are these things related?

Chapter 2

Limits and Colimits

There are two important ways to define new objects using ones you already have. The first of these is called taking the *limit* of a diagram – a limit is defined by its properties as the target of morphisms. The dual is the *colimit*, which is defined in terms of its properties as the domain of morphisms.

2.1 Diagrams and Their Shapes

We begin our discussion by revisiting diagrams from our more sophisticated category theoretical point of view. Let's first say that two diagrams have the same **shape** if one is obtained from the other by simply reassigning the objects and morphisms, but leaving the overall picture the same. For example, the diagrams

$$\begin{tikzcd} X \arrow[r, "f"] & Y & Z \arrow[l, "g"'] \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} A \arrow[r, "i"] & B & C \arrow[l, "j"'] \end{tikzcd}$$

in the category $\mathcal{C}$ are two different diagrams with the same shape, namely $\bullet \to \star \leftarrow \circ$. This last diagram could be called the common **shape diagram** for the two example diagrams.

Now, we have seen in Exercise 1.1 that we can, without causing any trouble, take any diagram and augment it by including all composite arrows and identity arrows that are not already present in the diagram, and the result is a category. If we apply this construction to the shape diagram $\bullet \to \star \leftarrow \circ$, we obtain a category $\mathcal{I}$. The key observation is that the diagram $\begin{tikzcd} X \arrow[r, "f"] & Y & Z \arrow[l, "g"'] \end{tikzcd}$ determines a functor

$$F : \mathcal{I} \to \mathcal{C}$$

given explicitly by

$$\begin{array}{rcl@{\qquad\qquad}rcl} F(\bullet) & = & X & F(\bullet \to \star) & = & f \\ F(\star) & = & Y & F(\star \leftarrow \circ) & = & g \\ F(\circ) & = & Z, & & & \end{array}$$

and of course F carries identities to identities. Now we can give our formal definition of a diagram.

Definition 14 Let $\mathcal{I}$ and $\mathcal{C}$ be two categories. A **diagram** in $\mathcal{C}$ with **shape** $\mathcal{I}$ is a (covariant) functor $F : \mathcal{I} \to \mathcal{C}$.

When $\mathrm{ob}(\mathcal{I})$ is too large to be a set, serious set-theoretical questions arise when working with $\mathcal{I}$-shaped diagrams. Therefore, the usual practice is to work only with **small diagrams** – diagrams whose shape 'only' has a set of objects. In this book we will *never* work with large diagrams.

Exercise 2.1 Determine the shape of the diagram[1]

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & B \arrow[d] \\ & C \end{tikzcd}$$

if the diagram is

(a) **guaranteed** to be commutative;

(b) **not necessarily** noncommutative.

Exercise 2.2 Let G be a group, and consider it as a category with one object, as in Exercise 1.3. Interpret 'diagram with shape G' in

(a) The category of sets and maps.

(b) The category of groups and homomorphisms.

(c) The category of topological spaces and continuous maps.

We call a shape category $\mathcal{I}$ **finite** if there is some $N \in \mathbb{N}$ such that in any composition $f_1 \circ f_2 \circ \cdots \circ f_m$ with $m > N$, at least one of the f_i is an identity map.

Exercise 2.3 Consider the group $G = \mathbb{Z}/2$ as a category with one object and two morphisms. Is G a small shape category?

We will want to study the collection of *all* diagrams with a given shape $\mathcal{I}$. To do this, we form the **diagram category**

$$\mathcal{C}^{\mathcal{I}} = \{\text{functors } \mathcal{I} \to \mathcal{C}\}$$

---

[1] in the sense of Section 1.1

The morphisms in this category are (of course) natural transformations between functors. How many natural transformations are there from one functor to another?

Exercise 2.4

(a) Let $\mathcal{I}$ be a small category and let $F, G \in \mathcal{C}^{\mathcal{I}}$. Express the collection of all natural transformations from F to G is a subset of some set. It follows that this collection is a set.

(b) Conclude that if $\mathcal{I}$ is small, then $\mathcal{C}^{\mathcal{I}}$ is actually a category.

(c) Find an example of a 'large' category $\mathcal{I}$ and $F, G \in \mathcal{C}^{\mathcal{I}}$ so that the collection of natural transformations $F \to G$ is not a set.

2.2 Limits and Colimits

Let's start with the diagram $X \to Y \leftarrow Z$ of the previous section. It frequently happens that we are given a diagram of this form, and we are interested in finding objects Q with maps $Q \to X$ and $Q \to Z$ such that the diagram

$$\begin{tikzcd}
Q \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& & X \arrow[d] \\
& Z \arrow[r] & Y
\end{tikzcd}$$

commutes. A particular example $P \to X$ and $P \to Z$ is a **limit** for the diagram if, for each other example $Q \to X$ and $Q \to Z$, there is a *unique* map $Q \to P$ making the diagram

$$\begin{tikzcd}
Q \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & X \arrow[d] \\
& Z \arrow[r] & Y
\end{tikzcd}$$

commute. That is, the limit of the diagram is any object (with morphisms to X and Z) that solves the universal problem posed by the diagram above.

Exercise 2.5 Let $\mathcal{U}$ be the category of whose objects are objects $Q \in \mathcal{C}$ together with maps $Q \to X$ and $Q \to Z$ making the diagram above commute; the morphisms are maps $f : Q \to R$ such that the diagram

$$\begin{tikzcd}
X \arrow[d, equal] & Q \arrow[l] \arrow[r] \arrow[d, "f"] & Z \arrow[d, equal] \\
X \arrow[dr, bend right] & R \arrow[l] \arrow[r] & Z \arrow[dl, bend left] \\
 & Y &
\end{tikzcd}$$

commutes. Show that P (with the maps to X and Z) is a limit for the diagram $X \to Y \leftarrow Z$ if and only if it is an terminal object in $\mathcal{U}$.

We can recast this definition in another way. Any object $Q \in \mathcal{C}$ gives rise to a constant functor

$$\underline{Q} : \mathcal{I} \to \mathcal{C}$$

given by $\underline{Q}$(any object) $= Q$ and $\underline{Q}$(any morphism) $= \mathrm{id}_Q$. This defines a functor

$$\Delta_{\mathcal{I}} : \mathcal{C} \to \mathcal{C}^{\mathcal{I}} \qquad \text{given by} \quad \Delta_{\mathcal{I}}(Q) = \underline{Q},$$

which is known as the **diagonal functor**. In our example, the commutativity of the square diagram is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
Q \arrow[r, equal] \arrow[d] & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] \\
X \arrow[r] & Y & Z \arrow[l]
\end{tikzcd}$$

or, in other words, the existence of a morphism (natural transformation) $\Delta_Q \to F$ in the category $\mathcal{C}^{\mathcal{I}}$. If P is the limit of the diagram, then we have a commutative diagram

$$\begin{tikzcd}
\underline{Q} \arrow[d, rightsquigarrow] & & Q \arrow[r, equal] \arrow[d] & Q \arrow[r, equal] \arrow[d] & Q \arrow[d] \\
\underline{P} \arrow[d, rightsquigarrow] & & P \arrow[r, equal] \arrow[d] & P \arrow[r, equal] \arrow[d] & P \arrow[d] \\
F & & X \arrow[r] & Y & Z. \arrow[l]
\end{tikzcd}$$

This leads to the following general definition.

Definition 15 Let $F : \mathcal{I} \to \mathcal{C}$ be a functor. A **limit** for F is an object $P \in \mathcal{C}$ and a natural transformation $\underline{P} \to F$ such that for any other natural transformation $\underline{Q} \to F$, there is a unique map $f : Q \to P$ making the diagram

$$\begin{tikzcd}
& \underline{P} \arrow[d] \\
\underline{Q} \arrow[r] \arrow[ur, dotted, bend left, "\underline{f}"] & F
\end{tikzcd}$$

commutative.

It is very common to write 'P is a limit for the diagram F' and make no mention at all of the transformation $\underline{P} \to F$.

Exercise 2.6 Consider the diagram category $\mathcal{I}$ with two objects $\bullet$ and $\star$, with only identity morphisms. Let $\mathcal{C}$ denote the category of real vector spaces and linear transformations, and let $F(\bullet) = 0$ and $f(\star) = \mathbb{R}$.

(a) Show that the limit of $F : \mathcal{I} \to \mathcal{C}$ is $\mathbb{R}$; what is the natural transformation $\Delta_{\mathbb{R}} \to F$?

(b) Let V be any nontrivial vector space. Show that there are uncountably many natural transformations $T : \Delta_V \to \Delta_{\mathbb{R}}$ making the diagram

$$\begin{tikzcd}
& \Delta_{\mathbb{R}} \arrow[d] \\
\Delta_V \arrow[r] \arrow[ur, dotted, bend left, "T"] & F
\end{tikzcd}$$

commutative.

(c) Show that exactly one of these is of the form Δ_f for some linear transformation $f : V \to \mathbb{R}$.

Problem 2.7 Let P be a limit for the diagram $F : \mathcal{I} \to \mathcal{C}$ (so we are given a natural transformation $L : \Delta_P \to F$). Show that there is an isomorphism

$$\mathrm{mor}_{\mathcal{C}^{\mathcal{I}}}(\Delta_Q, F) \cong \mathrm{mor}_{\mathcal{C}}(Q, P).$$

In some categories, every diagram of this form has a limit, but in other categories (including some that are central to homotopy theory) limits need not always exist.

Exercise 2.8 Suppose that the empty diagram $\varnothing$ has a limit. Show that it is a terminal object in $\mathcal{C}$. Find a way to view a product as the limit of a diagram.

Exercise 2.9 Construct an example of a category and a diagram which has no limit.

Exercise 2.10 A **discrete category** is one in which the only morphisms are identities. What is the limit of a diagram whose shape is discrete?

Problem 2.11 Dualize the discussion above: define the colimit of a diagram $F : \mathcal{I} \to \mathcal{C}$.

kind of big to be left to the reader

Problem 2.12 Show that any two limits of a given diagram $F : \mathcal{I} \to \mathcal{C}$ are equivalent in $\mathcal{C}$. Also prove the dual statement about the uniqueness of colimits.

Hint Let P and Q be two limits of F; use the universal property of the limit to find maps $P \to Q$ and $Q \to P$. Alternatively, construct a natural isomorphism between the functors $\mathrm{mor}_{\mathcal{C}}(\,?\,, P)$ and $\mathrm{mor}_{\mathcal{C}}(\,?\,, Q)$.

Domains and Targets. Colimits C are defined in such a way that we are given information about morphisms $C \to Z$; so colimits are *domain-type* constructions. Dually, limits are defined so that we have information about morphims $Z \to L$; and hence limits are *target-type* constructions.

Exercise 2.13 Show that the colimit of the empty diagram $\varnothing \to \mathcal{C}$ is an initial object in $\mathcal{C}$. Show that the limit of the identity $\mathrm{id}_{\mathcal{C}}$ is an initial object in $\mathcal{C}$.

Exercise 2.14 Suppose $\mathcal{I}$ has an initial object, $\varnothing$, and let $F : \mathcal{I} \to \mathcal{C}$. Show that F has a limit. State and prove the dual result.

Exercise 2.15 What is the colimit of a diagram whose shape is discrete?

2.3 Naturality of Limits and Colimits

Let $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{I} \to \mathcal{C}$ be two diagrams and let $\Phi : F \to G$ be a natural transformation between them; and suppose that both diagrams have colimits in $\mathcal{C}$, call them X and Y, respectively. Then the definition implies that we have the following picture

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & G \arrow[r] & \Delta_Y \\
\Delta_X \arrow[urr, dotted, "\exists\,!\,\Delta_f"'] & &
\end{tikzcd}$$

for a uniquely determined morphism $f : X \to Y$. We call the unique map $f : X \to Y$ determined by this diagram the map of colimits **induced** by the map of diagrams.

Problem 2.16 Explain how a natural transformation of diagrams induces a map between the limits of those diagrams.

PROBLEM 2.17 Let $\mathcal{C}$ be a category in which every diagram with shape $\mathcal{I}$ has a colimit. For each $F \in \mathcal{C}^{\mathcal{I}}$, choose, once and for all, a colimit for F, and call it colim F.[2]

(a) Show that the assignment $F \mapsto \operatorname{colim} F$ defines a functor $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$.

(b) Show that the functors $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$ and $\mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ given by

$$F \mapsto \operatorname{colim} F \qquad \text{and} \qquad X \mapsto \Delta_X$$

are adjoint to one another. Which is the right adjoint and which is the left adjoint?

You have proved the following theorem.

Theorem 16 *Let $\mathcal{C}$ be a category. Then the following are equivalent:*

1. *Every $\mathcal{I}$-shaped diagram in $\mathcal{C}$ has a colimit.*
2. *The functor $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ has a left adjoint.*

EXERCISE 2.18 Formulate the dual of Theorem 16. Do you need to prove it? Explain.

Theorem 17 *Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be an adjoint pair of functors.*

(a) Suppose that every diagram in $\mathcal{C}^{\mathcal{I}}$ has a colimit. Then for every $F \in \mathcal{C}^{\mathcal{I}}$, $L(\operatorname{colim} F)$ is a colimit for $L \circ F \in \mathcal{D}^{\mathcal{I}}$.

(b) Suppose that every diagram in $\mathcal{D}^{\mathcal{I}}$ has a limit. Then for every $F \in \mathcal{D}^{\mathcal{I}}$, $R(\lim F)$ is a limit for $R \circ F \in \mathcal{C}^{\mathcal{I}}$.

PROBLEM 2.19 Prove Theorem 17.

2.4 Special Kinds of Limits and Colimits

Some diagram shapes are particularly useful, and their limits and colimits have special names.

2.4.1 Pullback

We began our discussion of limits with a particularly simple example. This example is extremely important, and it has a special name – the pullback. In our more efficient terminology, the **pullback** of a diagram $X \to Y \leftarrow Z$ is the limit of the corresponding diagram (functor) $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is the shape category $\bullet \to \star \leftarrow \circ$ (we will call any diagram with this shape a

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[2] We consider the object colim F as coming with structure maps to the diagram F.

prepullback diagram). If P is a pullback for the diagram $X \to Y \leftarrow Z$, then the structure maps for P and the given diagram fit together into the commutative square

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & X \arrow[d] \\
Z \arrow[r] & Y
\end{tikzcd}$$

which we call a **pullback square**.

PROBLEM 2.20 Suppose that

$$\begin{tikzcd}
P \arrow[r, "p"] \arrow[d, "i"'] & X \arrow[d, "j"] \\
Z \arrow[r, "z"] & Y
\end{tikzcd}$$

is a pullback square. Show that if j is an equivalence, then so is i.

HINT Use the map $j^{-1} \circ z$ to find a map $K : Z \to P$.

EXERCISE 2.21 Suppose that $\mathcal{C}$ has a terminal object τ. Show that products of X and Y are the same as pullbacks for the diagram $X \to \tau \leftarrow Y$.

Let's look at some specific examples.

PROBLEM 2.22 Show that in the category of sets and functions

$$\begin{tikzcd}
\{(a,b) \mid f(a) = g(b)\} \arrow[r, "\mathrm{pr}_A"] \arrow[d, "\mathrm{pr}_B"'] & A \arrow[d, "f"] \\
B \arrow[r, "g"] & C
\end{tikzcd}$$

is a pullback square. What is the pullback of $A \to * \leftarrow B$?

EXERCISE 2.23

(a) Consider the category whose objects are the integers $1, 2, 3, \ldots$ and with arrows given by divisibility. Give number-theoretical descriptions of pullbacks in this category.

(b) Repeat (a) but using the category whose objects are real numbers and whose morphisms correspond to inequalities $x \leq y$.

EXERCISE 2.24 Determine the pullback of the diagram

$$\begin{tikzcd}
\{1\} \arrow[r] & B & A \arrow[l, "f"']
\end{tikzcd}$$

in the category of groups and homomorphisms.

There is another kind of limit, which is closely related to pullback. It arises when you are given two morphisms $f, g : X \to Y$, and you want to study morphisms $w : W \to X$ such that $f \circ w = g \circ w$. An **equalizer** for f and g is a morphism $e : E \to X$ such that $f \circ e = g \circ e$, and for any other such map $w : W \to X$, there is a unique map $W \to E$ as in the diagram

$$\begin{tikzcd}
W \arrow[d, dotted, "\exists!"'] \arrow[dr, "w"] & & \\
E \arrow[r, "e"] & X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & Y
\end{tikzcd}$$

Exercise 2.25

(a) What is the shape category for the equalizer?

(b) If products exist in our category, then we can form the diagram

$$\begin{tikzcd}
Q \arrow[r] \arrow[d, "q"'] & Y \arrow[d, "\Delta"] \\
X \arrow[r, "(f,g)"] & Y \times Y.
\end{tikzcd}$$

Show that this is a pullback square if and only if Q (with given map $q : Q \to X$) is an equalizer for f and g.

(c) Show that if pullbacks always exist in $\mathcal{C}$, then so do equalizers. Is the converse true?

2.4.2 Pushout

The pushout is dual to the pullback. More precisely, a pushout is a colimit of a **prepushout** diagram – i.e., a diagram with shape $\bullet \leftarrow \star \to \circ$. A **pushout square** is a commutative square

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[d] \\
X \arrow[r] & P
\end{tikzcd}$$

in which the objcet P (together with the maps $Z \to P$ and $X \to P$) is a colimit of the diagram $X \leftarrow Y \to Z$.

Problem 2.26 State and prove the dual of 2.20.

Exercise 2.27 Work in the category **Sets** of all sets and all functions between sets. Suppose that $A \subseteq B$ and $A \subseteq C$, and that $i : A \to C$ and $p : A \to B$ are inclusion functions.

(a) Determine the pushout of the diagram $B \xleftarrow{p} A \xrightarrow{i} C$.

(b) Let $*$ denote a one-point set. Determine the pushout of the diagram $B \xleftarrow{p} A \xrightarrow{i} *$.

PROBLEM 2.28 Let X be a topological space, and suppose $X = A \cup B$, where $A, B \subseteq X$ are closed subspaces. Show that the diagram

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & X
\end{tikzcd}$$

is a pushout square.

EXERCISE 2.29 Suppose $\mathcal{C}$ has an initial object, and let $X, Y \in \mathcal{C}$. Show that the pushouts of the diagram $X \leftarrow \iota \rightarrow Y$ are the same as coproducts $X \sqcup Y$.

EXERCISE 2.30

(a) Consider the category whose objects are the integers $1, 2, 3, \ldots$ and with arrows given by divisibility. Give number-theoretical descriptions of pushouts in this category.

(b) Repeat (a) but using the category whose objects are real numbers and whose morphisms correspond to inequalities $x \leq y$.

EXERCISE 2.31 Determine the pushout of the diagram

$$\{1\} \longleftarrow B \xrightarrow{f} A$$

in the category of groups and homomorphisms. Compare with Exercise 2.24; what should the object you constructed here be called?

PROBLEM 2.32 Define the dual notion of **coequalizer**, and compare coequalizers with pushouts.

2.4.3 Telescopes and Towers

Telescopes and towers are the diagrams that generalize infinite ascending unions and infinite nested intersections.

A **telescope** diagram is a diagram of the form

$$X_1 \to X_2 \to \cdots \to X_n \to \cdots.$$

To view this kind of diagram formally – as a functor from a shape category – let $\mathbb{N}$ be the category with objects $\{1, 2, \ldots\}$ and with a unique morphism $i \to j$ if $i \leq j$ and no morphism at all if $i > j$. Then The rule $i \mapsto X_i$ is the object part of a functor $\mathbb{N} \to \mathcal{C}$.

The colimit of such a diagram is often referred to as the **direct limit** of the diagram. Roughly speaking, it is the object X_∞ which belongs at the 'end' of the sequence.

For the dual, we reverse the arrows and look at

$$Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots .$$

Formally, this is a functor $F : \mathbb{N}^{\mathrm{op}} \to \mathcal{C}$. We will call such a diagram a **tower**; a limit for a tower is frequently called an **inverse limit**.

EXERCISE 2.33 Reformulate the definition of direct and inverse limits in terms of commutative diagrams.

PROBLEM 2.34 Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \cdots$ be a telescope diagram. We define the **shift map**

$$\mathrm{shift} : \coprod_{n \geq 0} X_n \to \coprod_{n \geq 1} X_n$$

by the formula shift $= (\mathrm{in}_2 \circ f_1, \mathrm{in}_3 \circ f_2, \ldots, \mathrm{in}_{n+1} \circ f_n, \ldots)$. Show that in the pushout square

$$\begin{tikzcd}
(\coprod X_n) \sqcup (\coprod X_n) \arrow[r, "(\mathrm{shift},\mathrm{id})"] \arrow[d, "\nabla"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod X_n \arrow[d] \\
\coprod X_n \arrow[r] & P
\end{tikzcd}$$

the pushout P is a direct limit for the telescope. Formulate the dual result.

Problem 2.34 shows that colimits of telescopes can be constructed if you can construct infinite coproducts and pushouts. (If you look carefully, you will see that the pushout diagram here is secretly a coequalizer diagram.). It is possible to show that in any category $\mathcal{C}$, if you can construct arbitrary coproducts and coequalizers, then you can construct all colimits.

PROBLEM 2.35 Let $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} \quad X_{n+1} \xrightarrow{f_{n+1}} \cdots$ be a telescope diagram, and let X_∞ its colimit. If $k \leq l$, write $f_{k,l}$ for the unique map in this diagram from X_k to X_l. Let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function and set up the commutative ladder

$$\begin{tikzcd}
\cdots \arrow[r, "f_{n-1}"] & X_n \arrow[r, "f_n"] \arrow[d, "f_{n,r(n)}"'] & X_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "f_{n+1,r(n+1)}"] & \cdots \\
\cdots \arrow[r, "f_{r(n-1),r(n)}"] & X_{r(n)} \arrow[r, "f_{r(n),r(n+1)}"] & X_{r(n+1)} \arrow[r, "f_{r(n+1),r(n+2)}"] & \cdots .
\end{tikzcd}$$

Show that X_∞ is also the colimit of the bottom row, and that the induced map $X_\infty \to X_\infty$ is the identity map.

Exercise 2.36 Generalize the result of Problem 2.35 to other diagram shapes.

2.5 Formal Properties of Pushout and Pullback Squares

We conclude with some formal properties of pushout and pullback squares. These rules, and their homotopy-theoretical analogs, will be crucial throughout our study of homotopy theory.

Problem 2.37 Consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r] & D.
\end{tikzcd}$$

This is the dual of an earlier problem.

(a) Suppose the diagram is a pushout and that f is an equivalence. Show that g is also an equivalence.

(b) Suppose f and g are both equivalences. Show that the square is a pushout.

(c) State and prove the duals of (a) and (b).

Theorem 18 *Consider the diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\textcircled{I}"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\textcircled{II}"] & A_3 \arrow[d, "h_3"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3,
\end{tikzcd}$$

and denote the outside square by (T)*.*

(a) If (I) *and* (II) *are pushouts, then* (T) *is also a pushout.*

(b) If (I) *and* (T) *are pushouts, then* (II) *is also a pushout.*

Problem 2.38 Prove Theorem 18.[3]

The dual statements are also true, of course.

Theorem 19 *Consider the diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\textcircled{I}"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\textcircled{II}"] & A_3 \arrow[d, "h_3"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3,
\end{tikzcd}$$

---

[3] Should break into steps.

and denote the outside square by (T)*.*

(a) *If* (I) *and* (II) *are pullbacks, then* (T) *is also a pullback.*

(b) *If* (II) *and* (T) *are pullbacks, then* (I) *is also a pushout.*

We end the chapter with our first investigation into diagrams more complicated than pushouts and pullbacks. Consider the diagram

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] \arrow[r] & A_3 & & & A \\
B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] \arrow[rrr, squiggly, "\text{pushout}"] & & & B \arrow[u] \arrow[d] \\
C_1 & C_2 \arrow[l] \arrow[r] & C_3 & & & C \\
 & & & & & D \\
 & & & & \textcircled{?} \arrow[ur, dotted] \arrow[dl, dotted] & \\
X & Y \arrow[l] \arrow[r] & Z & W & &
\end{tikzcd}$$

Taking pushouts of the rows gives a prepushout diagram $A \leftarrow B \rightarrow C$. Taking pushouts of the columns gives a prepushout diagram $X \leftarrow Y \rightarrow Z$. Let D and Z be the respective pushouts. How do they compare?

Theorem 20 *The pushouts* D *and* W *are both colimits for the whole* 3×3 *diagram, and hence are equivalent to each other.*

PROBLEM 2.39 Prove Theorem 20

We will also need the dual result, which concerns the limits of diagrams of the form

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[d] & A_3 \arrow[l] \arrow[d] & & A \\
B_1 \arrow[r] & B_2 & B_3 \arrow[l] \arrow[rr, rightsquigarrow, "\text{pullback}"] & & B \arrow[u] \arrow[d] \\
C_1 \arrow[r] \arrow[u] & C_2 \arrow[u] & C_3 \arrow[l] \arrow[u] & & C \\
 & & & & D \\
X \arrow[r] & Y & Z \arrow[l] & W \arrow[ur, dotted, leftrightarrow, "\textcircled{?}" description] &
\end{tikzcd}$$

In this case, the pullbacks of the rows form a prepullback diagram $C \to A \leftarrow B$, and we let D be the pullback; likewise, we define W to be the pullback of $X \to Y \leftarrow Z$.

Theorem 21 *The pullbacks D and W are both limits for the whole 3×3 diagram, and hence are equivalent to each other.*

Theorem 20 has a vast generalization. Suppose we have a diagram

$$F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}.$$

If we fix $j \in \mathcal{J}$ we obtain a $F_j : \mathcal{I} \to \mathcal{C}$ given by $F_j(i) = F(i, j)$; and these diagrams are related by natural transformations corresponding to the morphisms in $\mathcal{J}$. Forming colimits over $\mathcal{I}$ yielding a new diagram

$$\operatorname{colim}_{\mathcal{I}} F : \mathcal{J} \to \mathcal{C},$$

and the colimit of this new diagram is an object $\operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F \in \mathcal{C}$.

Theorem 22 *For any $F : \mathcal{I} \times \mathcal{J} \to \mathcal{C}$,*

(a) $\operatorname{colim}_{\mathcal{J}} \operatorname{colim}_{\mathcal{I}} F \cong \operatorname{colim} F \cong \operatorname{colim}_{\mathcal{I}} \operatorname{colim}_{\mathcal{J}} F$, *and*

(b) $\lim_{\mathcal{J}} \lim_{\mathcal{I}} F \cong \lim F \cong \lim_{\mathcal{I}} \lim_{\mathcal{J}} F$.

PROBLEM 2.40 Prove Theorem 22.

Part I

Semi-Formal Homotopy Theory

Chapter 3

Categories for Topology

In topology, it is often true that theorems are true for 'most' spaces, but that they fail for certain off-the-wall spaces. This can lead to a bewildering proliferation of hypotheses. Our solution to this annoyance will be to work entirely inside a 'convenient' category of topological spaces in which we can apply our basic constructions without fear. Of course, this category cannot possibly contain all spaces, but we would like it to contain the 'vast majority' of all spaces; in particular, it is crucial that our category of spaces contain all CW complexes. Furthermore, it should be closed under formation of limits and colimits, and certain fundamental constructions involving mapping spaces should behave well in our category.

3.1 Spheres and Disks

We begin with spheres and disks. The n-dimensional **sphere** is the space

$$S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

This standard sphere is of course homeomorphic to any subspace of $\mathbb{R}^{n+1}$ having the form $\{x \in \mathbb{R}^{n+1} \mid |x - a| = r\}$ for any fixed center point a and radius $r > 0$. (Exercise *Write down the homeomorphism!*). The n-sphere is the boundary of the $(n+1)$-dimensional **disk**

$$D^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}.$$

The **northern hemisphere** of S^n is the set

$$D^n_N = \{x \in \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } x_{n+1} \geq 0\}$$

and the **southern hemisphere** is

$$D^n_S = \{x \in \mathbb{R}^{n+1} \mid |x| = 1 \text{ and } x_{n+1} \leq 0\}.$$

The function $j_N : D^n \to D^n_N$ given by

$$j_N : (x_1, x_2, \ldots, x_n) \mapsto \left(x_1, x_2, \ldots, x_n, \sqrt{1 - (x_1^2 + \cdots + x_n^2)}\right)$$

is a homeomorphism; similarly the formula

$$j_S : (x_1, x_2, \ldots, x_n) \mapsto \left(x_1, x_2, \ldots, x_n, -\sqrt{1 - (x_1^2 + \cdots + x_n^2)}\right)$$

defines a homeomorphism $j_S : D^n \to D^n_S$.

Problem 3.1 Let $i : S^n \hookrightarrow D^{n+1}$ be the inclusion of the boundary. Show that

$$\begin{tikzcd}
S^n \arrow[r, hook, "i"] \arrow[d, hook, "i"'] & D^{n+1} \arrow[d, "j_N"] \\
D^{n+1} \arrow[r, "j_S"] & S^{n+1}
\end{tikzcd}$$

is a pushout square.

The **unreduced suspension** of a space X is the space $\Sigma_0 X$ which is obtained from $X \times I$ by collapsing the $X \times \{0\}$ to a single point $[0]$ and also collapsing $X \times \{1\}$ to a single point $[1]$.

Problem 3.2 Show that $\Sigma_0 S^n \cong S^{n+1}$ for each n.

Hint Draw a picture with both $S^n \times I$ and S^{n+1} for the cases $n = 0, 1$.

Let $X = S^n - \{N\}$, where $N = (0, 0, \ldots, 0, 1) \in S^n$ is the **north pole** of the sphere. We can define a function $\sigma : X \to \mathbb{R}^n$ by the following rule:

1. if $x \in X$, then $x \neq N$, and there is a unique line ℓ joining x and N;
2. since the point x is 'lower' than the point N, the line ℓ will cross the plane $\mathbb{R}^n \subseteq \mathbb{R}^{n+1}$ (in exactly one point);
3. call this point you just found $\sigma(x)$.

This function σ is usually called **stereographic projection**. The amazingly useful property of this map is that it is a *homeomorphism* from $S^n - \{N\} \to \mathbb{R}^n$.

Problem 3.3 Prove that $\sigma : S^n - \{N\} \to \mathbb{R}^n$ is a homeomorphism. More generally, let K be any space homeomorphic to S^n, and let $x \in K$. Show that $K - \{x\} \cong \mathbb{R}^n$.

3.2 CW Complexes

The spaces we will primarily be concerned with are called CW complexes; these spaces are built, step by step, from spheres and disks. We will show later that, for our purposes, there is almost no loss of generality in restricting our attention to CW complexes. But for now, we aim to establish the formalities of their step by step construction, provide some important examples, and derive some useful basic results about them.

Definition 23 To begin with, we say that any *discrete* space X is a **CW complex** with **dimension zero**. Inductively, suppose X_n is a CW complex with dimension at most n. Let $\alpha_n : \coprod_k S^n \to X_n$ be any map from a disjoint union of copies of S^n to X_n (possibly an empty disjoint union!), and define X_{n+1} to be the pushout

$$\begin{tikzcd}
\coprod_k S^n \arrow[r, "\coprod_k i"] \arrow[d, "f"'] & \coprod_k D^{n+1} \arrow[d] \\
X_n \arrow[r, "j_n"] & X_{n+1}
\end{tikzcd}$$

where $i : S^n \hookrightarrow D^{n+1}$ is the inclusion of the boundary of the disk. Then X_{n+1}, or anything homeomorphic to X_{n+1}, is a **CW complex** with **dimension** at most $n+1$. This inductively defines what we mean by a finite-dimensional **CW complex**. The images of the interiors of the disks used to construct X are called the **cells** of X; the image of an (open) n-dimensional disk is called an n-cell.[1]

Suppose we apply this construction infinitely many times, yielding a sequence of CW complexes $X_0, X_1, \ldots, X_n, \ldots$ and maps $j_n : X_n \to X_{n+1}$ between them. Then we may form the diagram

$$\begin{tikzcd}
X_0 \arrow[r, "j_0"] & X_1 \arrow[r, "j_1"] & \cdots \arrow[r, "j_{n-1}"] & X_n \arrow[r, "j_n"] & X_{n+1} \arrow[r, "j_{n+1}"] & \cdots.
\end{tikzcd}$$

Let X be the colimit of this diagram (i.e., the union of the X_n); then X, or anything homeomorphic to X, is called a **CW complex**.

The space $X_n \subseteq X$ is called the n-**skeleton** of X

A **subcomplex** of a CW complex X is a subspace $K \subseteq X$ which is a CW complex constructed by using some, but not necessarily all, of the cells used to construct X. For example, each skeleton $X_n \subseteq X$ is a subcomplex of X.

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[1] Note that the 0-dimensional disk D^0, which is a single point, is its own interior!

PROBLEM 3.4 Show that if X is a CW complex and $A \subseteq X$ is a subcomplex, then X/A inherits the structure of a CW complex from the quotient map $X \to X/A$.

EXERCISE 3.5 Show that $\mathbb{R}$ is a CW complex.[2]

EXERCISE 3.6 Using Problem 3.1, prove that S^{n+1} is a CW complex with two cells of each dimension $k \leq n+1$. Also find a CW decomposition of S^n with a grand total of two cells.

Exercise 3.6 shows that two CW complexes that are different in the sense that the list of attaching maps are not exactly the same can be homeomorphic to the same space X. We refer to these different CW complexes as a **CW structure** on X or as a **CW decomposition** of X. We will see that it is often helpful to choose a CW structure that is well suited to the work at hand.

Real Projective Spaces. Define an action of the multiplicative group $\mathbb{Z}/2 = \{+1, -1\}$ on the space S^n by simple coordinatewise multiplication. Then we may form the quotient space

$$\mathbb{R}\mathrm{P}^n = S^n/(\mathbb{Z}/2).$$

More explicitly, $\mathbb{R}\mathrm{P}^n$ is the result of identifying every point $x \in S^n$ with its **antipodal** point $-x \in S^n$. The space $\mathbb{R}\mathrm{P}^n$ is called the n-dimensional (real) **projective space**.

EXERCISE 3.7 Using Exercise 3.6, show that $\mathbb{R}\mathrm{P}^n$ is a CW complex. What is the $(n-1)$-skeleton? What is the attaching map of the n-cell onto the $(n-1)$-skeleton?

HINT Work by induction. Every point x in the northern hemisphere is equivalent to exactly one point in the southern hemisphere.

Another Description of CW Complexes. Each n-cell of a CW complex X has a corresponding map $\chi : D^n \to X$ defined by the diagram

$$\begin{tikzcd}
& D^n \arrow[d, hook] \arrow[ddr, bend left, "\chi"] & \\
\coprod_k S^n \arrow[r, "\coprod_k i"] \arrow[d, "f"'] & \coprod_k D^{n+1} \arrow[d] & \\
X_n \arrow[r, "j_n"] & X_{n+1} \arrow[r] & X.
\end{tikzcd}$$

This is called the **characteristic map** of the cell.

---

[2] Jason Trowbridge's idea.

Exercise 3.8 Let X be a CW complex.

(a) Show that $\chi|_{\text{int}(D^n)} : \text{int}(D^n) \to X$ is an embedding – i.e., it is a homeomorphism onto its image. The image of χ is called an **open n-cell** of X.

(b) Show that X is the union of the interiors of its cells (we interpret the interior of a 0-cell to be the cell itself), and that those open cells are pairwise disjoint.

(c) Show that X has the unique largest topology so that all of the characteristic maps $\chi : D^n \to X$ are continuous.

This leads us to our point-set topological description of CW complexes.

Definition 24 A space X is a CW complex if it is a union $X = \bigcup_{i\in\mathcal{I}} C_i(X)$ where each $C_i(X) \cong \text{int}(D^{n_i})$ and

1. The closure of each $C_i(X) \cong \text{int}(D^{n_i})$ is contained in the union of finitely many cells, and

2. X has the largest topology so that all of the inclusions $C_i(X) \hookrightarrow X$ are continuous.

CW complexes have some convenient topological features.

Problem 3.9 Let X be a CW complex with cells $C_i(X)$.

(a) Show that a set $A \subseteq X$ is closed if and only if each intersection $A\cap C_i(X)$ is closed in $C_i(X)$.

(b) Suppose $A \subseteq X$ and for each open cell $\text{int}(C_i(X))$, $A \cap \text{int}(C_i(X))$ is either empty or a singleton. Show that A is closed.

(c) Show that a compact subset of X must be contained in a finite subcomplex of X.

(d) Show that a function $f : X \to Y$ is continuous if and only if its restriction to each cell $C_i(X)$ is continuous.

Problem 3.10 Show that if $A \subseteq X$ is a subcomplex, then X/A, with the usual quotient topology, inherits the structure of a CW complex from X. Explain in detail how the cells of X/A are related to the cells of X.

Products of CW Complexes. If the open cells $C_i(X)$ of X are indexed by $\mathcal{I}$ and the open cells $C_j(Y)$ of Y are indexed by $\mathcal{J}$, we have for each $i \in \mathcal{I}$ and $j \in \mathcal{J}$ the composition

$$C_i(X) \times C_j(Y) \xrightarrow{\chi_i\times\chi_j} X \times Y.$$

Problem 3.11

(a) Suppose $C_i(X) \cong \operatorname{int}(D^n)$ and $C_j(Y) \cong \operatorname{int}(D^m)$. Show that $C_i(X) \times C_j(Y) \cong \operatorname{int}(D^{n+m})$.

(b) Show that the inclusion $C_i(X) \times C_j(X) \hookrightarrow X \times Y$ is continuous.

(c) Show that $X \times Y$ is the union of the product cells $C_i(X) \times C_j(Y)$.

This problem suggests that the set equation

$$X \times Y = \bigcup_{\mathcal{I}\times\mathcal{J}} C_i(X) \times C_j(Y)$$

might actually a representation of $X \times Y$ as a CW complex. In any case the set $X \times Y$ has a unique topology which makes it into a CW complex with the given cells – let us call that space the **CW product** of X and Y, which we will denote $X \times_{\mathrm{CW}} Y$. Is the CW product $X \times_{\mathrm{CW}} Y$ of X and Y homeomorphic to the ordinary categorical product of X and Y?

We will address this question later, in Section 3.4.

3.3 Mapping Spaces

Some of the main properties that we require involve the space of maps from one space to another, so let's begin by establishing our notation for mapping spaces. Let's say X and Y are two topological spaces. Then we can study the set

$$\operatorname{map}(X, Y) = \{f : X \to Y \mid f \text{ is continuous}\}$$

of all continuous maps from X to Y. We give this set the compact-open topology.[3]

A great deal of the structure of mapping spaces is revealed by the study of three very important kinds of maps between them. Like any collection of functions, $\operatorname{map}(X, Y)$ automatically comes with an **evaluation map**

$$@ : \operatorname{map}(X, Y) \times X \to Y$$

given by the formula $(f, x) \mapsto f(x)$. Function composition defines a function

$$\circ : \operatorname{map}(X, Y) \times \operatorname{map}(Y, Z) \to \operatorname{map}(X, Z)$$

explicitly given by $(f, g) \mapsto g \circ f$. Finally, we have a bijection

$$\alpha : \operatorname{map}(X \times Y, Z) \to \operatorname{map}(X, \operatorname{map}(Y, Z))$$

---

[3] Actually, if X and Y are horrible spaces we are forced to use a modification of the compact-open topology. Fortunately, we will sweep these details under the rug.

given by the formula $\alpha(f) : x \mapsto \boxed{y \mapsto f(x,y)}$.

PROBLEM 3.12 Prove that α is a bijection.

EXERCISE 3.13 Show that $\mathrm{map}(\{x\}, Y) \cong Y$ for any space Y.

3.4 Topological Spaces

We will need to talk about pointed topological spaces and unpointed topological spaces; sometimes we will prove results that work equally well in both contexts. Therefore, we will use the following notation:

1. $\mathcal{T}_\circ$ will denote our category of unpointed spaces,
2. $\mathcal{T}_*$ is our category of pointed spaces,[4] and
3. $\mathcal{T}$ is either one.

The following theorem is the main result of [???], and it asserts that there is a category which is good enough for our work. The proof belongs to point-set topology, so we will simply take this theorem for granted.

Theorem 25 *There is a category $\mathcal{T}_\circ$ whose objects are topological spaces,*[5] *which contains every locally compact space, and which has the following properties for any $X, Y, Z \in \mathcal{T}_\circ$*

(a) $\mathrm{map}(X, Y) \in \mathcal{T}_\circ$

(b) The evaluation map $@ : \mathrm{map}(X, Y) \times X \to Y$ *is continuous.*

(c) The composition map $\circ : \mathrm{map}(X, Y) \times \mathrm{map}(Y, Z) \to \mathrm{map}(X, Z)$ *is continuous*

(d) The function $\alpha : \mathrm{map}(X \times Y, Z) \to \mathrm{map}(X, \mathrm{map}(Y, Z))$ *is a homeomorphism.*

(e) The colimit of any diagram of spaces in $\mathcal{T}_\circ$ is again a space in $\mathcal{T}_\circ$.

(f) The limit of any diagram of spaces in $\mathcal{T}_\circ$ is again a space in $\mathcal{T}_\circ$.

(g) If $X, Y \in \mathcal{T}$ are CW complexes, then the CW product $X \times_{\mathrm{CW}} Y$ is equal to the categorical product $X \times Y$.

In view of Theorem 25(g), we never again have to use the notation$X \times_{\mathrm{CW}} Y$. It is worth noting that if one of the CW complexes is finite, or if both of

---

[4] which will be defined in Section 3.6

[5] But not *all* topological spaces!

the CW complexes has only finitely many cells in each dimension, then the products are the same even for the whole category of topological spaces.

The following lemma is frequently useful in cell-by-cell construction of homotopies.

Lemma 26 *Let X be a CW complex, and give I the standard CW deomposition with two zero cells and one 1-cell. Show that in the CW product decomposition,*

$$(X \times I)_{n+1} \subseteq (X \times 0) \cup (X_n \times I) \cup (X \times 1) \subseteq X \times I \subseteq X \times I.$$

Problem 3.14 Prove Lemma 26.

The fact that $\operatorname{map}(X \times Y, Z) \cong \operatorname{map}(X, \operatorname{map}(Y, Z))$ is known as the **exponential law**. The reason for this terminology lies in an alternative notation for mapping spaces: $\operatorname{map}(X, Y) = X^Y$.[6] Using this notation, the exponential law reads $Z^{X \times Y} = (Z^Y)^X$. This rule plays a crucial role throughout homotopy theory.

Problem 3.15

(a) Show that $S^n, D^n \in \mathcal{T}_\circ$ for all $n \geq 0$.

(b) Show that all CW complexes are in $\mathcal{T}_\circ$.

Problem 3.16 Explicitly describe the CW structure of $X \times I$ in terms of the CW structure of X.

Problem 3.17 Suppose $A, X, Y \in \mathcal{T}_\circ$, and that $A \subseteq X$. Show that $X/A \in \mathcal{T}_\circ$ and $X \times Y \in \mathcal{T}_\circ$.

Problem 3.18 Let $X, Y, Z \in \mathcal{T}_\circ$.

(a) If $f : X \to Y$, then there is an 'induced function'

$$f^* : \operatorname{map}(Y, Z) \to \operatorname{map}(X, Z),$$

given by $f^*(g) = g \circ f$. Show that f^* is continuous.

(b) If $h : Y \to Z$, then there is an 'induced function'

$$h_* : \operatorname{map}(X, Y) \to \operatorname{map}(X, Z)$$

given by $h_*(g) = h \circ g$. Show that h_* is continuous.

Hint Express the function you are interested in as a composition of the map $\circ$ with another function.

---

[6] Indeed, in the category of sets and their functions, this is how Cantor defined exponentiation of transfinite cardinals!

Problem 3.19

(a) Define two functors $F, G : \mathcal{T}_\circ \to \mathcal{T}_\circ$ by the rules

$$F(\,?\,) = \mathrm{map}(\,?\, \times Y, Z) \quad \text{and} \quad G(\,?\,) = \mathrm{map}(\,?\,, \mathrm{map}(Y, Z)).$$

Show that the exponential law defines a natural equivalence between these two functors. That is, use the exponential law to define a natural transformation $\Phi : F \to G$ and demonstrate that for every space in $X \in \mathcal{T}_\circ$, Φ_X is an equivalence.

(b) Define functors $P, M : \mathcal{T}_\circ \to \mathcal{T}_\circ$ by the rules

$$P(\,?\,) = \,?\, \times Y \quad \text{and} \quad M(\,?\,) = \mathrm{map}(Y, \,?\,).$$

Show P and M are adjoint functors.

3.5 The Category of Pairs

Suppose $A \subseteq X$ and $B \subseteq Y$. We will sometimes find ourselves interested *only* in those maps $f : X \to Y$ with the additional property that $f(A) \subseteq B$. These can be written as commutative diagrams

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y, \end{tikzcd}$$

but they are more typically written in the form

$$f : (X, A) \to (Y, B),$$

and (X, A) and (Y, B) are called **pairs** of spaces. The set

$$\mathrm{map}((X, A), (Y, B)) = \{f : (X, A) \to (Y, B)\} = \{f : X \to Y \mid f(A) \subseteq B\}$$

is clearly a subset of $\mathrm{map}(X, Y)$; thus it inherits a topology from $\mathrm{map}(X, Y)$. There is a category of pairs, which we will denote $\mathcal{T}_{(2)}$. There is an inclusion of categories $\mathcal{T}_\circ \to \mathcal{T}_{(2)}$ given by $X \mapsto (X, \varnothing)$.

Corollary 27 *The space* $\mathrm{map}((X, A), (Y, B))$ *is in* $\mathcal{T}_\circ$.

Problem 3.20 Prove Corollary 27 by showing that

$$\begin{tikzcd} \mathrm{map}((X, A), (Y, B)) \arrow[r] \arrow[d] & \mathrm{map}(X, Y) \arrow[d] \\ \mathrm{map}(A, B) \arrow[r] & \mathrm{map}(A, Y) \end{tikzcd}$$

is a pullback square.

Any map $f : X \to Y$ such that $f(X) \subseteq B$ is automatically an element of $\mathrm{map}((X,A),(Y,B))$; the space of all such maps is of course (almost) identically equal to $\mathrm{map}(X,B)$, and we won't belabor the distinction.[7] Thus, we consider space of maps $\mathrm{map}((X,A),(Y,B))$ as the *pair*

$$\left(\underbrace{\mathrm{map}((X,A),(Y,B))}_{\text{big space}}, \ \underbrace{\mathrm{map}(X,B)}_{\text{subspace}} \right).$$

The product of two pairs (X,A) and (Y,B) in $\mathcal{T}$ is also a pair in $\mathcal{T}$:

$$(X,A) \times (Y,B) = \left(\underbrace{X \times Y}_{\text{big space}}, \ \underbrace{A \times Y \cup X \times B}_{\text{subspace}} \right).$$

With these preliminaries, you can generalize some of the results of Theorem 25 to maps of pairs.

Problem 3.21

(a) Show that the exponential law holds for maps of pairs.
(b) Show that the exponential law is a natural isomorphism for pairs.
(c) Show that the composition function $\circ$ is well-defined and continuous for maps of pairs.

3.6 Pointed Spaces

Far and away, the most important special case for us is the one in which A is just a single point of X and B is a single point of Y; these special points are called **basepoints**. We usually denote the basepoint by $*$, no matter what space it is in. Maps of the form $f : (X,*) \to (Y,*)$ are called **pointed maps**. When we work with pointed spaces, we will usually suppress the pair notation. Thus, we will simply write X for a pointed space, and we will all know that it has a basepoint, and that basepoint is denoted $*$. We write $\mathcal{T}_*$ for the category of all pointed spaces $(X,*)$ such that the unpointed space $X \in \mathcal{T}_\circ$.

Products of Pointed Spaces. If we use the construction from the previous section, we find that the product of a pair of pointed spaces is not a pointed space:

$$(X,*) \times (Y,*) = (X \times Y, X \times * \cup * \times Y).$$

---

[7] Exercise What is the distinction?

The second term of this pair is the space obtained from the disjoint union of X and Y by identifying their basepoints; it is called the **wedge** of X and Y and is denoted $X \vee Y$. We are led to two related, but very different, kinds of products for pointed spaces.

1. The categorical product of pointed spaces X and Y is the ordinary product $X \times Y$, with basepoint $* \times *$.

2. The **smash product** of pointed space X and Y is the result of collapsing the wedge to a point: $X \wedge Y = (X \times Y)/(X \vee Y)$.

Problem 3.22

(a) Show that $X \vee Y$ is the categorical sum of X and Y in the category $\mathcal{T}_*$.[8]

(b) Show that the pointed space $X \times Y$ is the categorical product of X and Y in the category $\mathcal{T}_*$.

(c) Show that the rule $X \mapsto X \wedge A$ is a functor. Show that the three-variable functors $(X \wedge Y) \wedge Z$ and $X \wedge (Y \wedge Z)$ are naturally equivalent.

When these constructions are applied to CW complexes, they return CW complexes. It is helpful to know how the ouput CW structures are related to those of the inputs.

Problem 3.23 Suppose X and Y are pointed CW complexes. Show that $X \vee Y$ is a subcomplex of $X \times Y$. Explicitly relate the cells of $X \vee Y$ and $X \wedge Y$ to the cells of X, Y and $X \times Y$.

Pointed Mapping Spaces. When we work with pointed spaces, we use a simplification of the mapping space notation: instead of $\mathrm{map}((X, *), (Y, *))$, we write

$$\mathrm{map}_*(X, Y) = \{f : (X, *) \to (Y, *)\}$$

and call it the **space of pointed maps** from X to Y.

Exercise 3.24 The mapping space $\mathrm{map}_*(X, Y) = \{f : (X, *) \to (Y, *)\}$ is again a pointed space; what is the basepoint?

There is a way to go from pairs to pointed spaces: simply collapse the subspace to a point – i.e., replace (X, A) with $(X/A, *)$. Notice that the natural quotient map $(X, A) \to (X/A, *)$ is a map of pairs.

---

[8] Some authors refer to $X \vee Y$ as the 'wedge product,' which is terrible and confusing. It should be called the **wedge sum**, or just the **wedge**.

Problem 3.25 Show that the induced map

$$q^* : \mathrm{map}_*(X/A, Y) \to \mathrm{map}((X, A), (Y, *)).$$

is a bijection by constructing an inverse function ρ. Also show that q^* is continuous.

The Category of Pointed Spaces. We want to say that the pointed version of Theorem 25 is valid in the category $\mathcal{T}_*$, but this requires some discussion. It is not hard to formulate and verify the pointed versions of this theorem, except for part (d). For this we have a natural homeomorphism

$$\alpha : \mathrm{map}((X, *) \times (Y, *), (Z, *)) \to \mathrm{map}((X, *), \mathrm{map}((Y, *), (Z, *))).$$

The problem we have is that the domain in the first mapping space is not a pointed space – it is the pair $(X \times Y, X \vee Y)$. The way out of this is to collapse the wedge $X \vee Y$ to a point and use Problem 3.25. This gives us a new map

$$\begin{tikzcd}
\mathrm{map}_*\,(X \wedge Y, Z) \arrow[d, "q^*"', "\cong"] \arrow[dr, "\widetilde{\alpha}"] & \\
\mathrm{map}((X, *) \times (Y, *), (Z, *)) \arrow[r, "\alpha", "\cong"'] & \mathrm{map}_*(X, \mathrm{map}_*(Y, Z)).
\end{tikzcd}$$

The pointed version of Theorem 25(c) asserts that the map

$$\widetilde{\alpha} : \mathrm{map}_*(X \wedge Y, Z) \to \mathrm{map}_*(X, \mathrm{map}_*(Y, Z))$$

is a homeomorphism. Now that we know what we mean, we can make the following assertion.

Theorem 28 *There is a category $\mathcal{T}_*$ whose objects are pointed topological spaces, which contains every locally compact pointed space, and which has the following properties for any $X, Y, Z \in \mathcal{T}_*$*

- *(a)* $\mathrm{map}_*(X, Y) \in \mathcal{T}_*$
- *(b) The evaluation map* $@ : \mathrm{map}_*(X, Y) \times X \to Y$ *is continuous.*
- *(c) The composition map* $\circ : \mathrm{map}_*(X, Y) \times \mathrm{map}_*(Y, Z) \to \mathrm{map}_*(X, Z)$ *is continuous*
- *(d) The function* $\alpha : \mathrm{map}_*(X \times Y, Z) \to \mathrm{map}_*(X, \mathrm{map}(Y, Z))$ *is a homeomorphism.*
- *(e) The colimit of any diagram of spaces in $\mathcal{T}_*$ is again a space in $\mathcal{T}_*$.*
- *(f) The limit of any diagram of spaces in $\mathcal{T}_*$ is again a space in $\mathcal{T}_\circ$.*

Exercise 3.26 Show that $\mathcal{T}_*$ is a pointed category.

Relating the Categories of Pointed and Unpointed Spaces. It sometimes happens that we have a problem in $\mathcal{T}_\circ$, and we would like to study it using functors that are defined on $\mathcal{T}_*$, or vice versa. To do this, we need to clarify the relationship between these two categories. The categories $\mathcal{T}_\circ$ and $\mathcal{T}_*$ are related to one another in two ways.

First of all, forgetting the basepoint of a pointed space is a functor $I : \mathcal{T}_* \to \mathcal{T}_\circ$. Going the other way, we can take an unpointed space X and form the space

$$X_+ = X \sqcup *, \qquad \text{with basepoint} \quad * \, .$$

PROBLEM 3.27 Show that the rule $X \mapsto (X_+, *)$ is part of a functor $Q : \mathcal{T}_\circ \to \mathcal{T}_*$, and that I and Q are an adjoint pair. Which is the left adjoint and which is the left adjoint?

EXERCISE 3.28 Show that the rule rule $(X, A) \mapsto (X/A, *)$ is part of a functor $\widetilde{Q} : \mathcal{T}_{(2)} \to \mathcal{T}_*$. We can imbed $\mathcal{T}_\circ$ into $\mathcal{T}_{(2)}$ by the functor $\mathcal{T}_\circ \to \mathcal{T}_{(2)}$ given by $X \mapsto (X, \varnothing)$. What does $\widetilde{Q}$ do to the unpointed space X?

Based CW Complexes. In based CW complexes, we usually assume that the base point is a vertex – a point of X_0. Then the CW skeleta X_n are also given the same basepoint, and so the inclusions $X_n \hookrightarrow X$ are pointed maps.

3.7 Constructions in $\mathcal{T}_*$

There is another kind of product, midway between the ordinary product and the smash product, which is useful enough to be given its own name and notation. It is called the **half-smash product** of $X, Y \in \mathcal{T}_*$ is

$$X \rtimes Y = \frac{X \times Y}{* \times Y}.$$

Notice that we can form the half-smash even when Y is not a pointed space!

EXERCISE 3.29 The constructions $X \wedge Y$ and $X \rtimes Y$ are functorial in both variables. Assuming X and Y are CW complexes, express $X \rtimes Y$ as a CW complex.

We defined the smash product in the last section, and now we study a particular case of considerable importance: the (reduced) **suspension** of X is $\Sigma X = S^1 \wedge X$.

EXERCISE 3.30

(a) Using the identification $S^1 \cong I/\{0,1\}$, show that

$$\Sigma X \cong \frac{X}{(X \times \{0\}) \cup (X \times \{1\}) \cup (\{*\} \times I)}.$$

(b) If X is a pointed space, then there is a copy of $I \subseteq \Sigma_0 X$, namely $* \times I$. Show that $\Sigma X \cong \Sigma_0 X/I$.

(c) Suppose X is a CW complex. Describe a CW decomposition of ΣX in terms of the given one for X.

We will often use this identification, because it gives us a handy notation for points in ΣX; a typical point can be written as $[x,t]$, the equivalence class of $(x,t) \in X \times I$.

EXERCISE 3.31 For this problem, we use the spheres $S^n \subseteq \mathbb{R}^{n+1}$ centered at $(\frac{1}{2}, 0, \ldots, 0)$, radius $\frac{1}{2}$ and basepoint at the origin $\mathbf{0}$.

(a) For $\mathbf{x} \in \mathbb{R}^n \times \{0\} \subseteq \mathbb{R}^{n+1}$ there is a well-defined circle in the hyperplane containing the points $\mathbf{0}, \mathbf{x}$ and $\mathbf{e}_{n+1}$ which has the segment joining $\mathbf{0}$ to $\mathbf{x}$ as a diameter. Find a constant speed parametrization $\alpha : I \longrightarrow \mathbb{R}^{n+1}$ of this circle.

(b) Define $\phi : \Sigma S^n \longrightarrow S^{n+1}$ by $\phi([\mathbf{x}, t]) = \alpha_{\mathbf{x}}(t)$. Show that ϕ is a homeomorphism $\Sigma S^n \cong S^{n+1}$.

HINT Draw the picture in the cases $n = 0$ and $n = 1$.

The (reduced) **cone** on a pointed space is the space $CX = X \wedge I$ (we'll use $1 \in I$ as the basepoint in this context). The (reduced) **cylinder** on X is the space $X \rtimes I$ (here the basepoint of I is immaterial). The cone CX comes with a natural inclusion map

$$i : X \hookrightarrow CX \quad \text{given by} \quad x \mapsto [x, 0].$$

There are two natural inclusions $i_0, i_1 : X \hookrightarrow X \rtimes I$, given by $i_0(x) = [x, 0]$ and $i_1(x) = [x, 1]$.

EXERCISE 3.32 Draw pictures of cylinders, cones, and the inclusion maps so that you understand why they are named as they are.

PROBLEM 3.33 Show that there is a pushout square

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \arrow[d] \\
CX \arrow[r] & \Sigma X.
\end{tikzcd}$$

It is up to you to precisely define the maps in this square.

Exercise 3.34 Let X be any space, and let $* \in X$. Determine the mapping spaces

$$\mathrm{map}(S^0, X) \quad \text{and} \quad \mathrm{map}_*(S^0, X).$$

In other words, give a complete description of these spaces in terms of X, and not including any mapping spaces.

A special case of pointed mapping spaces that is very important is the set of pointed maps from S^1 to X; this particular mapping space is denoted $\Omega(X)$, and is called the **loop space** of X.

Problem 3.35 According to your work in Problems 3.21 and 3.25,

$$\mathrm{map}_*(X, \Omega(Y)) \cong \mathrm{map}_*(\mathbf{S}, Y).$$

for some pointed space $\mathbf{S}$. Give an explicit description of the space $\mathbf{S}$. Use your answer to identify $\mathrm{map}_*(S^n, \Omega(S^m))$ with the space of maps between two familiar spaces.

Problem 3.36 Show that for $0 \leq k \leq n$, there is a homeomorphism

$$\mathrm{map}_*(S^n, X) \cong \Omega^k(\mathrm{map}_*(S^{n-k}, X)),$$

where Ω^k indicates the loop space functor has been applied k times.

3.8 Mixed Adjunctions

It sometimes happens that we have a problem in $\mathcal{T}_\circ$, and we would like to study it using functors that are defined on $\mathcal{T}_*$, or vice versa. To do this, we need to clarify the relationship between these two categories using the forgetful functor $I : \mathcal{T}_* \to \mathcal{T}_{(2)}$ and the quotient space functor $\widetilde{Q} : \mathcal{T}_{(2)} \to \mathcal{T}_*$.

Problem 3.37

(a) Work out $X_+ \wedge Y_+$ and $X \wedge Y_+$ in $\mathcal{T}_*$.

(b) Express $I(X \times Y)$ in terms of $I(X)$ and $I(Y)$. Also write out $Q(X \times Y)$ in terms of $Q(X)$ and $Q(Y)$.

(c) Repeat (c) for coproducts.

(d) Show that $\mathrm{map}_*(X_+, Y) \cong \mathrm{map}(X, Y)$ in $\mathcal{T}_*$.[9]

Problem 3.38 Show that the exponential law implies that there is a natural isomorphism

$$\mathrm{map}_*(X \rtimes Y, Z) \cong \mathrm{map}_*(X, \mathrm{map}(Y, Z))$$

for $X, Y, Z \in \mathcal{T}_*$. Conclude that the functors $? \rtimes Y$ and $\mathrm{map}(\,?\,, Z)$ are adjoint.

---

[9] If Y is a pointed space, then so is $\mathrm{map}(X, Y)$ for any (pointed or unpointed) space X; what is the basepoint?

Now we can derive some useful results about smashes and pushouts.

Proposition 29 *Consider the diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in either $\mathcal{T}$ or $\mathcal{T}_*$.

(a) *If the diagram is a pushout square, then so are*

$$\begin{tikzcd}
A \times X \arrow[r] \arrow[d] & B \times X \arrow[d] \\
C \times X \arrow[r] & D \times X
\end{tikzcd}, \quad
\begin{tikzcd}
A \rtimes X \arrow[r] \arrow[d] & B \rtimes X \arrow[d] \\
C \rtimes X \arrow[r] & D \rtimes X
\end{tikzcd} \quad \textit{and} \quad
\begin{tikzcd}
A \wedge X \arrow[r] \arrow[d] & B \wedge X \arrow[d] \\
C \wedge X \arrow[r] & D \wedge X
\end{tikzcd}$$

for any space X.

(b) *If the original diagram is a pullback square, then so are*

$$\begin{tikzcd}
\mathrm{map}(X, A) \arrow[r] \arrow[d] & \mathrm{map}(X, B) \arrow[d] \\
\mathrm{map}(X, C) \arrow[r] & \mathrm{map}(X, D)
\end{tikzcd} \quad \textit{and} \quad
\begin{tikzcd}
\mathrm{map}_*(X, A) \arrow[r] \arrow[d] & \mathrm{map}_*(X, A) \arrow[d] \\
\mathrm{map}_*(X, A) \arrow[r] & \mathrm{map}_*(X, A).
\end{tikzcd}$$

Problem 3.39 Prove Proposition 29 using Theorem 17

Problem 3.40 Show that for any $X \in \mathcal{T}_*$, $X_+ \simeq X \vee S^0$ in $\mathcal{T}$, but if $X \not\simeq *$, then $X_+ \not\simeq X \vee S^0$ in $\mathcal{T}_*$.

Corollary 30 *There is a natural equivalence*

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)$$

in the category $\mathcal{T}_*$.

Problem 3.41 Prove Corollary 30.

Chapter 4

Homotopy

Topology can be described as the study of all continuous maps between topological spaces, and the composition of those maps with one another. This is a huge problem because, for starters, there are generally going to be uncountably many maps from one space to another. The subject of homotopy theory arises when we simplify the grand problem by changing our object of study from continuous maps to equivalence classes of continuous functions under the equivalence relation called homotopy. This dramatically simplifies the situation and makes it possible to make some real progress.

4.1 Definition of Homotopy

Let's start by defining our equivalence relation. The intuitive idea is that we should consider f and g to be equivalent if f can be 'continuously deformed' into g. If we let this 'deformation occur between time $t = 0$ and $t = 1$, then we can let f_t be the map that results from the deformation up to time t; then $f_0 = f$ and $f_1 = g$.

While this point of view is frequently useful, it has proven to be easier to use a definition which views the rule $(x, t) \mapsto f_t(x)$ as a single function.

Definition 31 Two maps $f, g : X \to Y$ are **homotopic** if there is another map

$$H : X \times I \to Y \quad \text{such that} \quad H|_{X\times\{0\}} = f \quad \text{and} \quad H|_{X\times\{1\}} = g$$

The notation $f \simeq g$ denotes that f is homotopic to g. The map H is called a **homotopy**.

This first notion of homotopy does not require our deformation to respect any of the structure that X might have, apart from its topology. Such a homotopy is sometimes called a **free homotopy** when this needs emphasis.

Your first problem is to interpret homotopy in terms of mapping spaces; it is easier in the context of mapping spaces to find the 'correct' notion for homotopy of pointed maps.

Problem 4.1 Let X and Y be two spaces.

(a) Use the exponential law to interpret homotopy of maps from X to Y in terms of basic topological properties of the mapping space $\mathrm{map}(X, Y)$.

(b) Generalize the new interpretation to define **pointed homotopy**[1] of pointed maps $f, g \in \mathrm{map}_*(X, Y)$.

(c) Interpret your answer in (b) in the language of Definition 31.

(d) Show that pointed homotopies are in one-to-one correspondence with maps $H : X \rtimes I \to Y$.

Problem 4.2 Let X be a topological space, and let $x, y \in X$. Say that $x \sim y$ if x and y are in the same path component of X (i.e., if there is a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$).

(a) Show that $\sim$ is an equivalence relation.

(b) Conclude that $\simeq$ is an equivalence relation in both the free and pointed context.

We will almost always deal with pointed spaces, maps and homotopies in the book. But we do occasionally need to deal with free homotopy classes, so we'll establish different notation for the two cases. For any two spaces X and Y, we write

$$\langle X, Y \rangle = \{\text{free homotopy classes of maps } X \to Y\}$$

(there does not seem to be any standard notation for this set). If X and Y are pointed spaces, we write

$$[X, Y] = \{\text{pointed homotopy classes of maps } X \to Y\},$$

which is very much the standard notation. We denote the homotopy class of a map $f : X \to Y$ in $\mathcal{T}_\circ$ by $\langle f \rangle \in \langle X, Y \rangle$, and if $f \in \mathcal{T}_*$, then we write $[f] \in [X, Y]$.

Exercise 4.3 The set $[X, Y]$ is a pointed set – what is the basepoint?

---

[1] Pointed homotopy will also be denoted $f \simeq g$.

If X is a topological space, we denote by $\pi_0(X)$ the set of path components of X. In terms of *sets*, this is simply the set $X/\sim$, where $\sim$ is the equivalence relation you studied in Problem 4.2. If X is a pointed space, then $\pi_0(X)$ is a **pointed set** — there is a special point in $\pi_0(X)$, namely the equivalence class of the special point $* \in X$.

PROBLEM 4.4 We need to know that π_0 is functorial, so let $f : X \to Y$ and $g : Y \to Z$ be two maps in $\mathcal{T}_*$.

(a) Show that if $x, x' \in X$, then $f(x) \sim f(x')$ in Y.

(b) Define $\pi_0(f)$ – make sure that it is well defined.

(c) Verify that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$.

(d) Conclude that π_0 is a covariant functor from $\mathcal{T}_*$ to the category of pointed sets.

PROBLEM 4.5 Interpret $[X, Y]$ in terms of π_0. Is $[X, Y]$ functorial? Explain thoroughly, keeping in mind that there are two possible 'variables'.

HINT Think in terms of composition of functors.

PROBLEM 4.6 Let $f, g : X \to Y$, and suppose $f \simeq g$.

(a) Show that $f_*, g_* : \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$ are homotopic.

(b) Show that $f^*, g^* : \mathrm{map}(Y, B) \to \mathrm{map}(X, B)$ are homotopic.

(c) Prove the same thing for pointed mapping spaces.

HINT Use the interpretation of homotopies as paths in mapping spaces.

PROBLEM 4.7 Show that the functors $? \times Z$, $? \rtimes Z$ and $? \wedge Z$ also respect homotopy.

PROBLEM 4.8 Suppose $X \simeq Y$, and show that for any spaces A and B

(a) $\mathrm{map}_*(A, X) \simeq \mathrm{map}_*(A, Y)$ and $\mathrm{map}(A, X) \simeq \mathrm{map}(A, Y)$

(b) $A \wedge X \simeq A \wedge Y$, $A \rtimes X \simeq A \rtimes Y$, and $A \times X \simeq A \times Y$.

Let's conclude by looking at a specific example.

PROBLEM 4.9

(a) Let $x, y \in \mathbb{R}^n$. Write down a formula for the straight-line path from x to y.

(b) Let $X = \{*\}$, and let $f, g : X \to \mathbb{R}^n$ be defined by $f(*) = x$ and $g(*) = y$. Show that $f \simeq g$.

(c) Now let X be any space, and let $f, g : X \to \mathbb{R}^n$. Show that $f \simeq g$. What is $\langle X, \mathbb{R}^n \rangle$?

(d) Choose a point $a \in \mathbb{R}^n$, and let this be the basepoint of $\mathbb{R}^n$. Let X be a pointed space; what is $[X, \mathbb{R}^n]$?

Homotopies of Paths. A **path** in a space X is just a continuous function $\alpha : I \to X$. Homotopy classes of paths play an important role in many contexts, including complex analysis and geometry, not to mention homotopy theory.

Exercise 4.10 Show that if X is path connected, then any two paths are freely homotopic. If the paths are pointed, then they are pointed homotopic.

This problem seems to suggest that the homotopy theory of paths in spaces is entirely trivial. The content comes when we restrict our homotopies even further. If α and β are paths in X from x_0 to x_1, then a **path homotopy** $H : \alpha \simeq \beta$ is a is a homotopy H with the additional property that $H|_{0\times I}$ is constant at x_0 and $H|_{1\times I}$ is constant at x_1. In other words, each map $\alpha_t = H|_{I\times t}$ is a path from x_0 to x_1.

If $\alpha, \beta : I \to X$ are paths in X, then we define their concatenation

$$\alpha * \beta : I \to X$$

by the formula

$$\alpha * \beta(t) = \begin{cases} \alpha(2t) & \text{if } t \in [0, \frac{1}{2}] \\ \\ \beta(2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

We also define the reverse path $\overleftarrow{\alpha} : I \to X$ by $\overleftarrow{\alpha}(t) = \alpha(t-1)$.

Problem 4.11 Show that there is a path homotopy $\alpha * \overleftarrow{\alpha} \simeq *$, the constant path at $\alpha(0)$.

4.2 Composing and Inverting Homotopies

We compose two homotopies $H : f \simeq g$ and $K : g \simeq h$ by concatenating them like so

$$(H+K)(x,t) = \begin{cases} H(x, 2t) & \text{if } t \in [0, \frac{1}{2}] \\ \\ K(x, 2t-1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

There is a trivial homotopy, which we will refer to as the **static homotopy**, from $f : X \to Y$ to itself, namely $\textsc{static}_f : X \times I \to Y$ given by $\textsc{static}_f(x,t) = f(x)$.[2]

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[2] Thus $\textsc{static}_f = f \circ \mathrm{pr}_X$.

We can also form the **reverse** of a given homotopy, as you did in your proof that homotopy is a symmetric relation. If $H : X \times I \to Y$ is a homotopy, then its reverse is the homotopy $\overleftarrow{H} : X \times I \to Y$ given by $\overleftarrow{H}(x,t) = H(x, 1-t)$.

Two homotopies $H, K : X \times I \to Y$ are **homotopic** if there is a homotopy

$$J : (X \times I) \times I \to Y$$

which begins at H when $s = 0$ and ends at K when $s = 1$; and for every value of s, $J(x, 0, s) = f(x)$ and $J(x, 1, s) = g(x)$.

EXERCISE 4.12 In Problem 4.1 you showed how to interpret a homotopy of maps $X \to Y$ as a path in the mapping space $\mathrm{map}(X,Y)$. Show that in this interpretation,

(a) sum of homotopies is concatenation of paths,
(b) the reverse homotopy is just the reverse path, and
(c) a homotopy-of-homotopies is a path homotopy.

PROBLEM 4.13 Let $H : f \simeq g$ be a homotopy.

(a) Show that $H + \mathrm{STATIC}_g \simeq H \simeq \mathrm{STATIC}_f + H$.
(b) Show that there is a homotopy-of-homotopies from $H + \overleftarrow{H} \simeq \mathrm{STATIC}_f$.

HINT Think of the homotopies as paths in $\mathrm{map}(X,Y)$.

4.3 The Homotopy Category

We can make two new categories, called the pointed **homotopy category** and unpointed **homotopy category**. The pointed homotopy category is the most important for us, and we denote it by $\mathrm{H}\mathcal{T}_*$. The objects and morphisms of this category are

$$\begin{aligned} \mathrm{ob}(\mathrm{H}\mathcal{T}_*) &= \mathrm{ob}(\mathcal{T}_*) \\ \mathrm{mor}_{\mathrm{H}\mathcal{T}_*}(X,Y) &= [X,Y]. \end{aligned}$$

The unpointed homotopy category $\mathrm{H}\mathcal{T}_\circ$ is defined analogously; and we use $\mathrm{H}\mathcal{T}$ for statements that are equally valid in either category.

EXERCISE 4.14 Verify that $\mathrm{H}\mathcal{T}_*$ is actually a category, and that the obvious assignment $L : \mathcal{T}_* \to \mathrm{H}\mathcal{T}_*$ given by $L : X \mapsto X$ and $L : f \mapsto [f]$ (the pointed homotopy class of f) is a functor. Is it fair to call L a forgetful functor?

We say that a diagram $F : \mathcal{I} \to \mathcal{T}_*$ is **homotopy commutative** if it becomes commutative after applying the functor L – i.e., if the composite diagram $L \circ F : \mathcal{I} \to \mathrm{H}\mathcal{T}_*$ is commutative.

Exercise 4.15 Give an example of a diagram that is homotopy commutative but not commutative.

We will get in the habit of dealing with homotopy commutative diagrams; when we wish to be clear that a diagram is commutative (and not just homotopy commutative) we will say that it is **strictly commutative** or commutative 'on the nose.'[3]

Problem 4.16 Show that there is a natural equivalence

$$[X \wedge Y, Z] \cong [X, \mathrm{map}_*(Y, Z)].$$

Definition 32 Two spaces X and Y are (pointed) **homotopy equivalent** if they are equivalent in the homotopy category $\mathrm{H}\mathcal{T}$. This is denoted $X \simeq Y$; in the special case $X \simeq *$, X is called (pointed) **contractible**.

Exercise 4.17 Write out explicitly in terms of maps, spaces and homotopies exactly what it means for two spaces to be homotopy equivalent. Does it make a difference if you work with pointed or unpointed spaces – that is, could there be two pointed spaces which are homotopy equivalent in $\mathcal{T}_\circ$ but not in $\mathcal{T}_*$?

Problem 4.18 Show that the following are equivalent for $Y \in \mathcal{T}_*$.

1. $Y \simeq *$
2. $[X, Y] = *$ for all $X \in \mathcal{T}_*$
3. $[Y, Z] = *$ for all $Z \in \mathcal{T}_*$.

Problem 4.19

(a) Show that if $X \simeq *$, then X is path connected.

(b) Show that if $X \simeq *$, then $\Omega X \simeq *$ and $\Sigma X \simeq *$.

(c) Find an example of a space X such that $X \not\simeq *$ but its loop space $\Omega X \simeq *$.

Problem 4.20

(a) Show that $\mathbb{R}^n$ is contractible.

(b) Let $X \in \mathcal{T}_\circ$, and suppose $f : X \to S^n$ is map which is not surjective. Show that f is homotopic to a constant map – i.e., show that $f \simeq *$. Is this true in $\mathcal{T}_*$?

Nullhomotopic Maps. A map $f : X \to Y$ that is homotopic to the constant map $* : X \to Y$ is called a **nullhomotopic** map, or a **trivial** map. A homotopy $H : f \simeq *$ is sometimes called a **nullhomotopy**. There are several useful criterial for deciding if a map is nullhomotopic or not.

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[3] or, perhaps, 'nasally commutative'?

PROBLEM 4.21 Let $f : X \to Y$. Show that the following are equivalent:

1. f is nullhomotopic
2. there is a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "i"'] & & Y \\
& CX. \arrow[ur, "e"'] &
\end{tikzcd}$$

3. f factors (up to homotopy) through a contractible space.

PROBLEM 4.22 Show that a map $\alpha : S^n \to X$ is homotopic to $*$ if and only if it factors through D^{n+1} – that is, if and only if there is a map $\bar{\alpha} : D^{n+1} \to Y$ such that the diagram

$$\begin{tikzcd}
S^n \arrow[rr, "\alpha"] \arrow[dr, "i"'] & & Y \\
& D^{n+1} \arrow[ur, "\bar{\alpha}"'] &
\end{tikzcd}$$

is strictly commutative.

For $X \in \mathcal{T}_*$, we consider the space of pointed paths

$$\mathcal{P}(X) = \{\omega : I \to X \,|\, \omega(0) = *\},$$

and the evaluation map $@_1 : \mathcal{P}(X) \to X$ given by $@_1(\omega) = 1$.

PROBLEM 4.23 Show that $\mathcal{P}(X)$ is contractible.

PROBLEM 4.24 Let $f : X \to Y$ and show that the following are equivalent

1. f is nullhomotopic
2. there is a lift λ in the diagram

$$\begin{tikzcd}
& \mathcal{P}(Y) \arrow[d, "@_1"] \\
X \arrow[r, "f"] \arrow[ur, dotted, bend left, "\lambda"] & Y
\end{tikzcd}$$

Now we show how to correlate our two criteria. Let $f : X \to Y$. You have shown how to construct, from a nullhomotopy $H : f \simeq *$, maps $CX \to Y$ extending f and $X \to \mathcal{P}(Y)$ lifting X.

Problem 4.25 Show that there is a map $CX \to \mathcal{P}(X)$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "\lambda"] \arrow[d, "\mathrm{in}_1"'] & \mathcal{P}(Y) \arrow[d, "@_1"] \\
CX \arrow[r, "e"] \arrow[ur, dotted] & Y
\end{tikzcd}$$

strictly commutative.

Problem 4.26

(a) Show that there is a pullback square

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & \mathcal{P}(X) \arrow[d, "@_1"] \\
* \arrow[r] & X
\end{tikzcd}$$

(b) Construct the diagram

$$\begin{tikzcd}
\Omega X \arrow[r, "\mathrm{in}_0"] \arrow[d] & C\Omega X \arrow[r, "e"] \arrow[d] & \mathcal{P}(X) \arrow[d, "@_1"] \\
* \arrow[r] & \Sigma\Omega X \arrow[r, "L"] & X.
\end{tikzcd}$$

Write out explicit formulas for the maps e and L.

(c) The exponential law gives an explicit homeomorphism

$$\mathrm{map}_*(\Sigma\Omega X, X) \cong \mathrm{map}_*(\Omega X, \Omega X).$$

Show that under this bijection, L corresponds to the identity map $\mathrm{id}_{\Omega X} \in \mathrm{map}_*(\Omega X, \Omega X)$.

4.4 Groups and Cogroups in h$\mathcal{T}_*$

In order to get algebra involved in our topology, we need to find some cogroup objects in the category H$\mathcal{T}_*$.

Let's start by studying the space $S^1 \subseteq \mathbb{C}$; we will use the point $1 \in S^1$ as the basepoint $*$. If we identify the point -1 with $*$, the result is a wedge of two circles; thus the quotient map ϕ is a map

$$\phi : S^1 \to S^1 \vee S^1.$$

We can also reverse the orientation of the circle by flipping it over, giving a map $\nu : S^1 \to S^1$.

EXERCISE 4.27 Write down explicit formulas for the maps ϕ and ν in each of the following contexts:

(a) think of points in S^1 as complex numbers;

(b) think of points in S^1 as angles between 0 and 2π;

(c) think of S^1 as $I/\{0,1\}$.

The next theorem gives us the most important example of a cogroup object in the category $\mathrm{h}\mathcal{T}_*$ – the circle!

Theorem 33 *The maps ϕ and ν make S^1 into a cogroup object in $\mathrm{h}\mathcal{T}_*$.*

EXERCISE 4.28 Think of S^1 as the quotient space $I/\{0,1\}$, with basepoint $[0] = [1]$. Let $f : S^1 \to S^1$ be the map which collapses the (image of) the line segment $[a,b]$ to a point and stretches the rest of the circle uniformly, where $0 \leq a < b \leq 1$. Show that $f \simeq 1_{S^1}$. Can the homotopy be chosen to be a pointed homotopy?

PROBLEM 4.29 Prove Theorem 33. Is S^1 a cogroup object in $\mathcal{T}_\circ$?

HINT This ultimately boils down to showing that certain maps $S^1 \to S^1$ are homotopic. Exercise 4.28 can be useful.

Once we have one cogroup object in hand, we can construct many more, as the next problem shows.

PROBLEM 4.30

(a) If A is a cogroup object in $\mathrm{h}\mathcal{T}_*$, show $A \wedge X$ is also a cogroup. Describe the structure maps for $A \wedge X$ in terms of those for A. What can you say about $A \wedge X$ if A is a commutative cogroup?

HINT Use the functoriality of $? \wedge X$.

(b) Show that any space X, the suspension ΣX is a cogroup object in $\mathrm{h}\mathcal{T}_*$. Write down explicit formulas for $\alpha + \beta$ and $-\alpha$ for $\alpha, \beta \in [\Sigma X, Y]$.

(c) Show that S^n is a cogroup object in $\mathrm{h}\mathcal{T}_*$ for all $n \geq 1$.

PROBLEM 4.31

(a) Write down the composition of maps which defines $1_{S^1} \cdot 1_{S^1} \in [S^1, S^1]$. We'll refer to this map as $\mathbf{2} : S^1 \to S^1$.

(b) Write down the composition of maps which defines $f \cdot f \in [S^1, X]$.

(c) We have $\mathbf{2}^* : [S^1, X] \to [S^1, X]$. Show that $\mathbf{2}^*(f) = f \cdot f$.

Notice that Theorem 33 does not guarantee that $\mathbf{2}^*$ is a homomorphism, and in fact it won't be if $[S^1, X]$ is not abelian.

Abelian Cogroup Objects. Since abelian groups are so much easier to deal with than general groups, it will be very helpful to be able to recognize abelian cogroup objects. The most important example of an abelian cogroup object in $\text{h}\mathcal{T}_*$ is S^2.

Theorem 34 *The two-sphere S^2 is a cocommutative cogroup in the homotopy category.*

Problem 4.32 View $S^2 = \{x \in \mathcal{R}^3 \,|\, |x| = 1\}$ with basepoint $* = (1, 0)$. Let $R_\theta : S^2 \to S^2$ be rotation by θ radians about the x-axis (note that R_θ is a pointed map).

(a) Write down explicitly the formula for the pinch map $\phi : S^2 \to S^2 \vee S^2$ that collapses the equatorial circle $S^2 \cap (\mathbb{R}^2 \times 0)$ to a point.

(b) Show that $R_\theta \simeq \mathrm{id}_{S^2}$.

(c) Show that the diagram

$$\begin{tikzcd}
S^2 \arrow[r, "R_\pi"] \arrow[d, "\phi"'] & S^2 \arrow[d, "\phi"] \\
S^2 \vee S^2 \arrow[r, "T"] & S^2 \vee S^2
\end{tikzcd}$$

is strictly commutative.

(d) Prove Theorem 34.

Using Theorem 34 we get a vast collection of abelian cogroup objects in $\text{h}\mathcal{T}_*$.

Exercise 4.33 Show that for any space X, the double suspension $\Sigma^2 X$ is a cocommutative cogroup object in $\text{h}\mathcal{T}_*$. In particular S^n is a commutative cogroup for $n \geq 2$.

There are cocommutative cogroup objects that are not double suspensions, but even these are retracts of double suspensions.

Group Objects in h$\mathcal{T}_*$. We close this section with a brief survey of the dual results.

Problem 4.34 Let X be a cogroup object in $\mathcal{T}_*$.

(a) Show that $\mathrm{map}_*(X, Y)$ is a group object in $\mathcal{T}_*$ for any space Y, and if X is a cocommutative cogroup object, then $\mathrm{map}_*(X, Y)$ is a commutative group object.

(b) Show that if Y is a group object, then $\mathrm{map}_*(X, Y)$ is a group object, and if Y is commutative, then so is $\mathrm{map}_*(X, Y)$.

Theorem 35 *For any space X,*

(a) ΩX is a group object in $\mathrm{H}\mathcal{T}_*$*, and*

(b) $\Omega^2 X$ is a commutative group object in $\mathrm{H}\mathcal{T}_*$*.*

PROBLEM 4.35 Prove Theorem 35.

4.5 Homotopy Groups

Now we come to our first, and arguably most important, specific collection of functors – the **homotopy groups** of a pointed space $X \in \mathcal{T}_*$. For $n \geq 0$, we define

$$\pi_n(X) = [S^n, X],$$

and if $f : X \to Y$, then $\pi_n(f) = f_* : \pi_n(X) \to \pi_n(Y)$. We know by Proposition 4 that these rules do in fact define covariant functors. On the face of it, these functors take there values in the category of pointed sets, but the results of the last section imply that they can be given more algebraic structure.

Theorem 36

(a) For $n \geq 1$, the functor π_n takes its values in the category $\mathcal{G}$ of groups and homomorphisms.

(b) For $n \geq 2$, the functor π_n takes its values in the category $\mathrm{AB}\,\mathcal{G}$*.*

PROBLEM 4.36 Prove Theorem 36.

We now have two definitions of π_0 – the one defined above in terms of S^0 and the one defined in Section 3.1 using path components.

EXERCISE 4.37 Show that the two definitions of the functor π_0 are naturally equivalent.

EXERCISE 4.38 Intepret the groups $\pi_n(\mathrm{map}_*(X, Y))$ and $\pi_n(\mathrm{map}(X, Y))$ as sets of the form $[A, B]$, with no mapping spaces or πs involved.

EXERCISE 4.39 Show that for any space $X \in \mathcal{T}_*$, there is a wedge of spheres $W = \bigvee_\alpha S^{n_\alpha}$ and a map $w : W \to X$ such that the induced map

$$w_* : \pi_n(W) \to \pi_n(X)$$

is surjective for every n.

Many introductory topology classes finish by studying the **fundamental group** of a space, i.e., $\pi_1(X)$. Therefore, we will take the following theorem for granted.

Theorem 37 $\pi_1(S^1) \cong \mathbb{Z}$.

Proof Sketch. Think of $S^1 \subseteq \mathbb{C}$, and so every function $f : S^1 \to S^1$ can be thought of as a function $f : S^1 \to \mathbb{C}$ which does not hit the point 0. Thus we can compute the *winding number* of f around zero – denote this by $W(f)$ (in complex analysis, this is done by evaluating a certain line integral). If $f \simeq g$, then $W(f) = W(g)$, and so the winding number defines a function $W : \pi_1(S^1) \to \mathbb{Z}$. The map W is a group isomorphism. □

Let $f : X \to Y$ be a map in $\mathcal{T}_\circ$. If we want think of f as a pointed map, we can choose a basepoint $x_0 \in X$; but then in order for f to be a pointed map, we are forced to choose $y_0 = f(x_0)$ as the basepoint of Y. Let's call the resulting pointed map f_{x_0}. Thus we have a procedure which takes an unpointed map f and produces a huge collection of pointed maps, one for each point $x_0 \in X$.

4.6 Homotopy and Duality

As we progress, we will find ourselves proving various results about homotopies, and we will of course want to keep our eyes open for results that can be dualized. But what is the dual of a homotopy? That is the question we address in this section.

Let Y be a space, and consider the mapping space $Y^I = \mathrm{map}(I, Y)$. This is called the (unpointed) **path space** of Y. If you choose a point $t \in I$, then evaluation at t defines a function

$$@_t : Y^I \to Y.$$

Problem 4.40

(a) Show that $@_t : Y^I \to Y$ is a homotopy equivalence.

(b) Show that homotopies $H : X \times I \to Y$ from f to g correspond by the exponential law to functions $K : X \to Y^I$ such that $@_0 \circ K = f$ and $@_1 \circ K = g$.

Now we have two ways to talk about homotopies of maps from $X \to Y$; they can be thought of as maps

$$X \times I \to Y \qquad \text{or as maps} \qquad X \to Y^I.$$

The purpose of this discussion is to nail down carefully the sense in which these two notions of homotopy are dual to one another.

To achieve this, we add another layer of abstraction. For a space Y, we define a **path object** for Y to be a space, which we'll denote $\mathrm{Path}(Y)$ even though it is not unique, together with a map $p : \mathrm{Path}(Y) \to Y \times Y$. We'll write $p = (p_0, p_1)$, and both components are required to be homotopy equivalences. Here it is as a diagram

$$\begin{tikzcd}
 & \mathrm{Path}(Y) \arrow[dl, bend right, "p_0"', "\simeq"] \arrow[d, "p"] \arrow[dr, bend left, "p_1", "\simeq"'] & \\
Y & Y \times Y \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"'] & Y
\end{tikzcd}$$

The dual notion is that of a **cylinder object**, which should fit into the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "\simeq", "i_0"'] & X \sqcup X \arrow[d, "i"] & X \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "\simeq"', "i_1"] \\
 & \mathrm{Cyl}(X). &
\end{tikzcd}$$

Problem 4.41

(a) Show that Y^I, together with the maps $p_0 = @_0$ and $p_1 = @_1$, constitutes a path object for Y.

(b) Show that $X \times I$, together with the maps $i_0 = \mathrm{in}_0$ and $i_1 = \mathrm{in}_1$, constitutes a cylinder object for X.

Now we define a **left homotopy** to be a function $H : \mathrm{Cyl}(X) \to Y$ such that the following diagram commutes

$$\begin{tikzcd}
X \sqcup X \arrow[d, "(i_0,i_1)"'] \arrow[dr, bend left, "(f,g)"] & \\
\mathrm{Cyl}(X) \arrow[r, "H"] & Y,
\end{tikzcd}$$

where $\mathrm{Cyl}(X)$ is any cylinder object for X. Dually, a **right homotopy** is a map $K : X \to \mathrm{Path}(Y)$ such that

$$\begin{tikzcd}
 & Y \times Y \\
X \arrow[ur, bend left, "(f,g)"] \arrow[r, "K"] & \mathrm{Cyl}(Y) \arrow[u, "(p_0,p_1)"']
\end{tikzcd}$$

where $\mathrm{Path}(Y)$ is any path object for Y.

Exercise 4.42

(a) Show that if we choose $\mathrm{Cyl}(X) = X \times I$ with $i = (\mathrm{in}_0, \mathrm{in}_1)$, then a left homotopy is the same thing as an ordinary homotopy.

(b) Do the same for $\mathrm{Path}(Y) = Y^I$ with $p = (@_0, @_1)$.

Problem 4.43 Let $f, g : X \to Y$, and suppose you are given a path object and a cylinder object, not necessarily the standard ones. Show that if f and g are either right homotopic or left homotopic, then they are homotopic in the standard sense.

Unfortunately, the converse of this problem is not true with the definition we have given so far.

Problem 4.44

(a) Show that (with our definition), the space X, together with the folding map $\nabla : X \sqcup X \to X$, is a cylinder object for X. What does it take for two maps to be left homotopic using this particular cylinder object?

(b) Show that (with our definition), the space Y, together with the diagonal map $\Delta : Y \to Y \times Y$ is a path object for Y. What does it take for two maps to be right homotopic using this particular path object?

The solution to the difficulties pointed out in Problem 4.44 is to impose additional restrictions on the maps $p : \mathrm{Path}(Y) \to Y$ and $i : X \sqcup X \to X$. Once this is done it will turn out that ordinary homotopy of maps is equivalent to left homotopy or right homotopy, using any path or cylinder objects you choose. We will come back to these issues in Chapter **??**.

Chapter 5

Cofibrations and Fibrations

Certain maps are very well suited to working with homotopies. When we're talking about a map $i : A \to X$ of *domains*, the key question is whether or not it is possible to extend a homotopy of a map $A \to Y$ to a homotopy defined on X. Maps where this extension is always possible are called *cofibrations.* When we're concerned with targets, we say that a map $p : E \to B$ is a *fibration* if any homotopy of a map $X \to B$ can be lifted to a homotopy in E.

5.1 Cofibrations

5.1.1 Definition of Cofibrations

Let $A \subseteq X$ and $f : X \to Y$; then the restriction of f to A, $f|_A : A \to Y$, is also a map with target A. Suppose we are given a homotopy $H_A : A \times I \to Y$ from $f|_A$ to some other map $g_A : A \to Y$. An **extension** of the homotopy H_A to X is a homotopy $H : X \times I \to Y$ such that $H_A = H|_{A\times I}$. This is illustrated in the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i_0"'] & X \arrow[d, "i_0"'] \arrow[ddr, bend left, "f"] & \\
A \times I \arrow[r, "i\times \mathrm{id}_I"] \arrow[drr, bend right, "H_A"'] & X \times I \arrow[dr, dotted, "H"] & \\
 & & Y
\end{tikzcd}$$

For some particularly nice maps $i : A \to X$, *every* homotopy can be extended.

Definition 38 A map $i : A \to X$ has the **homotopy extension property** if, for any map $f : X \to Y$, and any homotopy $H_A : A \times I \to Y$ of $f|_A$ to some other map, there is an extension $H : X \times I \to Y$. If i has the homotopy extension property, then i is called a **cofibration**.

The homotopy extension property for the map $i : A \to X$ is frequently expressed in the diagram form

$$\begin{tikzcd}
A \arrow[r, "H_A"] \arrow[d, "i"'] & Y^I \arrow[d, "@_0"] \\
X \arrow[r, "f"] \arrow[ur, dotted, "H"] & Y.
\end{tikzcd}$$

The idea here is that you have a homotopy H_A that is define only on the subset $A \subseteq X$, and the question is whether or not that partial homotopy can be extended to a homotopy on all of X.

Problem 5.1 Show that the composition of two cofibrations is a cofibration.

Exercise 5.2 Let $i : A \to X$ be a cofibration.

(a) Show that any homeomorphism is a cofibration.

(b) Let $B = i(A) \subseteq X$ and let $j : A \to B$ be the function obtained from i by restricting the target. Show that j is a homeomorphism.

(c) Show there is an open set $U \subseteq X$ containing A such that A is a retract of U.

It follows from Exercise 5.2(b) that every cofibration can be expressed as a composition $A \to B \to X$, where $A \to B$ is a bijective, and $B \to X$ is the inclusion of a subspace. Problem 5.1, together with Exercise 5.2(a) implies that $A \to X$ is a cofibration if and only if $B \to X$ is a cofibration. However, you will show later that not every inclusion map is a cofibration. This all goes to show that it makes sense to think of cofibrations as 'well-behaved inclusion maps.'

One of the nice features of cofibrations is that they allow you to replace squares that are homotopy commutative with squares that are commutative 'on the nose.'

Problem 5.3 Let

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "g"] \\
C \arrow[r, "j"] & D
\end{tikzcd}$$

be a homotopy commutative square, and suppose that i is a cofibration. Show that there is a map $\gamma : B \to D$ such that $\gamma \simeq g$ and the diagram

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "\gamma"] \\
C \arrow[r, "j"] & D
\end{tikzcd}$$

is strictly commutative.

Cofibrations and Pointed Spaces. So far, we have been working with *unpointed* homotopies $A \times I \to X$, and you must have been asking 'what about cofibrations for pointed spaces?'

Exercise 5.4

(a) By replacing all free homotopies with pointed homotopies, formulate the definition of a pointed cofibration.

(b) Show that if $A \to X$ is a map in $\mathcal{T}_*$ which is an unpointed cofibration,[1] then it is also a pointed cofibration.

(c) Give an example of a map $A \to X$ in $\mathcal{T}_*$ which is a pointed cofibration but not an unpointed cofibration.

Because of parts (b) and (c) of the Exercise 5.4, we see that the notion of unpointed cofibration is strictly stronger than that of pointed fibration.

When working with pointed spaces, it is often helpful to know that the inclusion of the basepoint $* \hookrightarrow X$ is a cofibration (in the unpointed sense). When this is the case, we say that X is **well pointed**, or **cofibrant**.

Exercise 5.5 Give an example of a pointed space which is not cofibrant.

5.1.2 Reformulations

Next we derive a number of alternative descriptions of cofibrations. Each of them is useful in its own way.

The first of these arises by comparing $X \times I$ with a pushout.

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[1] more precisely, if $I : \mathcal{T}_* \to \mathcal{T}_\circ$ is the forgetful functor, then $I(A) \to I(X)$ is an cofibration as above.

Problem 5.6 Consider the diagram used to define cofibrations:

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "\mathrm{in}_0"'] & X \arrow[d, "\mathrm{in}_0"'] \arrow[ddr, bend left, "f"] & \\
A \times I \arrow[r, "i\times \mathrm{id}_I"] \arrow[drr, bend right, "H_A"'] & X \times I \arrow[dr, dotted, "H"] & \\
 & & Y.
\end{tikzcd}$$

This should remind you of the definition of the pushout. Let $T = X \cup_A (A \times I)$ be the pushout of the diagram $A \times I \xleftarrow{\mathrm{in}_0} A \xrightarrow{i} X$.[2] Show that $i : A \to X$ is a cofibration if and only if the inclusion $T \hookrightarrow X \times I$ has a retraction $r : X \times I \to T$.

We can say something even stronger about the retraction $r : X \times I \to T$, but we need some more terminology first.

If X is a retract of Y, then we have maps $i : X \to Y$ and $r : Y \to X$ such that $r \circ i = \mathrm{id}_X$. If, in addition, the composition $i \circ r : Y \to Y$ is homotopic to id_Y, then we say that X is a **deformation retract** of Y. If the homotopy $H : \mathrm{id}_Y \simeq i \circ r$ can be found such that $H|_{X\times I} = \mathrm{pr}_X$, then X is said to be a **strong deformation retract** of Y.

Proposition 39 *A map $i : A \to X$ is a cofibration if and only if the space $T = X \cup_A (A \times I)$ is a strong deformation retract of $X \times I$.*

problemize?

Proof. If T is a deformation retract, then it is an ordinary retract, so the 'if' implication follows immediately from Problem 5.6. For the converse, if i is a cofibration then Problem 5.6 gives a retraction $r : X \times I \to T$. Write r coordinatewise in the form $r(x,t) = (r_1(x,t), r_2(x,t))$, where $r_1(x,t) \in X$ and $r_2(x,t) \in I$. Inspired by the idea of the straight-line homotopy, we define $H : r \circ i \simeq \mathrm{id}_{X\times I}$ by the formula

$$H((x,t),s) = (r_1(x,st), (1-s)t + sr_2(x,t)).$$

You can easily check that this homotopy is constant on T. □

The following characterization of cofibrations comes uses the space $T = X \cup (A \times I)$ as a 'universal example' for the cofibration property. It is a universal example in the sense that if you can extend *this* homotopy, then you can extend *any* homotopy.

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[2] In particular, note that $X \times I$ is *not* the pushout of the square!

PROBLEM 5.7 Let $i : A \hookrightarrow X$, and let $T = X \cup_A (A \times I)$ as in Problem 5.6. Let $i_0 : X \to T$ be the inclusion of X at level $t = 0$, and let $i_1 : A \to T$ be the inclusion of A at level $t = 1$. Let $H : A \times I \to T$ be the inclusion function $(a, t) \mapsto (a, t)$. Show that i is a cofibration if and only if the homotopy H can be extended to a homotopy $\bar{H} : X \times I \to T$ from the inclusion $X \hookrightarrow T$ to some other map.

Let's look at some basic examples.

PROBLEM 5.8

(a) Show that the inclusion $A \xrightarrow{\mathrm{in}_0} A \times I$ is a cofibration.

HINT Use Problem 5.6.

(b) Show that the inclusion $i : S^n \hookrightarrow D^{n+1}$ is a cofibration.

HINT Use Problem 5.6. Do this first in the case $n = 0$; draw a picture including the point $(\mathbf{0}, 2) \in \mathbb{R}^{n+1} \times \mathbb{R}$, and think geometrically – there is no need to write down a complicated formula!

(c) Let $X = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$ and $A = \{0\}$. Show that $A \hookrightarrow X$ is not a cofibration.

(d) Show that the inclusion $i : \mathbb{Q} \hookrightarrow \mathbb{R}$ is not a cofibration.

HINT For (c) and (d), use Problem 5.7.

PROBLEM 5.9 Show that if $A \hookrightarrow X$ is a cofibration and X is Hausdorff, then A is closed in X. Show by example if X is not Hausdorff, then A need not be closed.

HINT Show that if A is not closed, then something like Problem 5.8(c) can happen.

A subset $A \subseteq X$ is a (strong) **neighborhood deformation retract** if there is an open set $U \subseteq X$ containing A and a strong deformation retraction $r : U \to A$. We show that $A \hookrightarrow X$ being a cofibration is almost equivalent to A being a neighborhood deformation retract.

Theorem 40 *The inclusion $A \hookrightarrow X$ of a closed subspace is a cofibration if and only*

1. *A is a strong neighborhood deformation retract of a neighborhood $U \subseteq X$ and*
2. *there is function $u : X \to I$ such that $A = u^{-1}(0)$ and $U = u^{-1}([0, 1))$.*

You should take this for granted.

5.2 Constructions and Examples

In this section, we'll develop some techniques for constructing new cofibrations out of old ones.

Pushouts and Cofibrations. We begin with one of the most crucial properties of cofibrations: their behavior with respect to pushouts.

Theorem 41 *Let $i : A \to X$ be a cofibration, and suppose the square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d, "j"] \\
X \arrow[r] & Y
\end{tikzcd}$$

is a pushout square. Then $j : B \to Y$ is a cofibration.

Problem 5.10 Prove Theorem 41 by studying the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & B \arrow[r] \arrow[d, "j"] & Z^I \arrow[d, "@_0"] \\
X \arrow[r] \arrow[urr, dotted] & Y \arrow[r] \arrow[ur, dotted] & Z.
\end{tikzcd}$$

Products and Cofibrations. We next turn to the maps that arise when we form the product of two pairs of spaces. If $A \to X$ and $B \to Y$ are cofibrations, the we have maps

$$A \times B \to (A \times Y) \cup (X \times B) \to X \times Y.$$

We show that these maps, and hence their composition, are cofibrations.

Proposition 42 *If $A \to X$ and $B \to Y$ are cofibrations, then*

$$A \times Y \cup X \times B \to X \times Y$$

is also a cofibration.

Proof. Let $u : X \to I$ and $H : X \times I \to X$ be the maps guaranteed by Theorem 40 because $A \hookrightarrow X$ is a cofibration, and likewise let and $v : Y \to I$ and $K : Y \times I \to Y$ be the maps which show that $B \hookrightarrow Y$ is a cofibration. Then the maps

$$\begin{aligned} w(x,y) &= \min\{u(x), v(y)\} \qquad \text{and} \\ J(x,y,t) &= \begin{cases} \left(H(x,t), K\left(y, t\frac{u(x)}{v(y)}\right)\right) & \text{if } v(y) \geq u(x) \\ \left(H\left(x, t\frac{v(y)}{u(x)}\right), K(y,t)\right) & \text{if } u(x) \geq v(y) \end{cases} \end{aligned}$$

satisfy the conditions of Theorem 40, and so prove the Proposition. □

Problem 5.11 Suppose $i : A \hookrightarrow X$ and $j : B \hookrightarrow Y$ are cofibrations.

(a) Show that $A \times Y \hookrightarrow X \times Y$ is a cofibration.

(b) Show that $A \times B \hookrightarrow (A \times Y) \cup (X \times B)$.

Hint The space $(A \times Y) \cup (X \times B)$ is a pushout. is a cofibration.

(c) Show that $A \times B \hookrightarrow X \times Y$ is a cofibration.

Problem 5.12 Suppose A and X are both cofibrant. Prove that if $A \hookrightarrow X$ is a pointed cofibration then it is also an unpointed cofibration.

Important Examples. Now we can find some of the most frequently used examples of cofibrations.

Problem 5.13

(a) Let K be a space, and let $X = K \cup_f D^n$, the space obtained from K by attaching a disk using the map $f : S^{n-1} \to K$. Show that the inclusion map $i : K \hookrightarrow X$ is a cofibration.

(b) Show that if X is a finite CW complex and $K \subseteq X$ is a subcomplex, then the inclusion $K \hookrightarrow X$ is a cofibration.

Problem 5.14 Let A and B be any two cofibrant spaces.

(a) Show that $A \vee B \hookrightarrow A \times B$ is a cofibration.

(b) Show that the inclusion $A \hookrightarrow A \vee B$ is a cofibration. Is it necessary that both spaces be well pointed?

(c) Show that $i_0 : A \to A \rtimes I$ is a cofibration.[3]

Hint Show that the square

$$\begin{tikzcd}
A \vee I \arrow[r, hook] \arrow[d, "(\mathrm{id}_A, *)"'] & A \times I \arrow[d, "q"] \\
A \arrow[r, "i_0"] & A \rtimes I
\end{tikzcd}$$

is a pushout square.

(d) Show that $(i_0, i_1) : A \vee A \to A \rtimes I$ is a cofibration.

(e) Recall that the cone on A is $CA = A \wedge I$. Show that $i_1 : A \hookrightarrow CA$ is a cofibration.

Hint Set up a pushout square similar to the one in part (c).

(f) Show that the inclusion $X \hookrightarrow X \cup_f CA$ is a cofibration, no matter what the attaching map $f : A \to X$ is.

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[3] Of course, exactly the same reasoning will lead you to the conclusion that i_1 is a cofibration.

5.3 Exact Sequences

Cofibrations are used to bring exact sequences into topology, and exact sequences are perhaps the most important tool that we have when it comes to making explicit calculations in homotopy theory.

A sequence of pointed sets $A \to B \to C$ is **exact** if the kernel of $B \to C$ (i.e., the set of all $b \in B$ that map to $* \in C$) is precisely equal to the image of $A \to B$. Most often we deal with exact sequences of *groups* — a group is a pointed set whose basepoint is the identity element. The most basic example of an exact sequence of groups is

$$N \to G \to G/N,$$

where N is a normal subgroup of G, and the maps are the ones you expect. There are sequences of this kind in topology: those of the form $A \to X \to X/A$. Should we hope that, for any space Y, the corresponding sequence

$$[X/A, Y] \longleftarrow [X, Y] \longleftarrow [A, Y]$$

of pointed sets is exact? It turns out that this is true as long as the inclusion $i : A \hookrightarrow X$ is a cofibration.

Theorem 43 *Let $i : A \to X$ be a cofibration in $\mathcal{T}_*$, and let $q : X \to X/A$ be the canonical quotient map. Then for any space $Y \in \mathcal{T}_*$, the sequence of pointed sets*

$$[A, Y] \xleftarrow{i^*} [X, Y] \xleftarrow{q^*} [X/A, Y]$$

is exact.

Problem 5.15 We continue to use the hypotheses and notation of Theorem 43.

(a) Let $f : X \to Y$ and assume that there is a map $g : X/A \to Y$ such that $g \circ q \simeq f$. Show that $f|_A \simeq *$.

(b) Let $f : X \to Y$, and suppose that $f|_A \simeq *$. Show that there is a map $g : X/A \to Y$ such that $g \circ q \simeq f$.

Hint First show that there is a map $k : X \to Y$ such that $k(A) = *$ and such that $k \simeq f$. Get the map g from the map k.

(c) Prove Theorem 43.

Proposition 44 *Suppose $A \to X$ is a cofibration and A is contractible. Then the quotient map $q : X \to X/A$ is a homotopy equivalence.*

lemize!

Proof. Since A is contractible, there is a homotopy $H : \mathrm{id}_A \simeq *$. Since $A \hookrightarrow X$ is a cofibration, we can extend H to a homotopy $J : X \times I \to X$ with $J(A \times 1) = *$. Define $j_t(x) = J(x,t)$, so $j_t : X \to X$ for each $t \in I$, $j_0 = \mathrm{id}_X$, and $j_1(A) = *$. Since $j_1(A) = *$, we can factor j_1 through X/A as in the picture

$$\begin{tikzcd}
X \arrow[rr, "j_1\simeq \mathrm{id}_X"] \arrow[dr, "q"'] & & X \\
& X/A. \arrow[ur, "\widetilde{j}_1"'] &
\end{tikzcd}$$

We claim that q and $\widetilde{j}_1$ are homotopy inverses, and we start by observing that the homotopy J gives

$$\widetilde{j}_1 \circ q = j_1 \simeq \mathrm{id}_X.$$

On the other hand, since $H(A,t) \subseteq A$ for each value of t, we can construct functions $\widetilde{j}_t : X/A \to X$ for each t, and the function

$$K : (X/A) \times I \to X/A$$

given by $K([x],t) = q(\widetilde{j}_t([x]))$ is a homotopy $K : q \circ \widetilde{j}_1 \simeq \mathrm{id}_{X/A}$. □

EXERCISE 5.16 Suppose $A \simeq *$ and $A \hookrightarrow X$ is a map in $\mathcal{T}_*$. Are X and X/A homotopy equivalent as pointed spaces, or just as free spaces?

We derive some instant corollaries.

EXERCISE 5.17 Prove that $X \rtimes I \simeq X \times I$ and $\Sigma X \simeq \Sigma_0 X$.

Together with Problem 3.2, this gives an easy proof that $\Sigma S^n \simeq S^{n+1}$, though we already know that ΣS^n is homeomorphic to S^{n+1}.

5.4 Every Map Is Equivalent to a Cofibration

One of the key facts that makes homotopy theory work is that every map $f : X \to Y$ is 'equivalent to' a cofibration. To be more precise about this, we will use the following definition.

Definition 45 Two maps $f : A \to B$ and $g : X \to Y$ are **homotopy equivalent** if there is a homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"'] & B \arrow[d, "b"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}$$

where the vertical arrows a and b are homotopy equivalences. [4]

Exercise 5.18

(a) Compare this definition to our previous definition for equivalence of maps in Exercise 1.7.

(b) Show that if $f \simeq g : X \to Y$, then f and g are homotopy equivalent.

(c) Generalize this definition to one for homotopy equivalent diagrams. Can you think of variations on the idea of homotopy equivalent diagrams or maps? We'll come back to this in much more detail in the next chapter.

The nice thing about homotopy equivalent maps is that they produce 'the same' map on sets of homotopy classes.

Problem 5.19 Suppose $f : A \to B$ and $g : X \to Y$ are homotopy equivalent maps in $\mathcal{T}_*$. Let Z be any space, and consider the diagrams

$$\begin{tikzcd}
{[A,Z]} & {[B,Z]} \arrow[l, "f^*"'] \\
{[X,Z]} \arrow[u, "a^*"] & {[Y,Z]} \arrow[l, "g^*"] \arrow[u, "b^*"']
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
{[Z,A]} \arrow[r, "f_*"] \arrow[d, "a_*"'] & {[Z,B]} \arrow[d, "b_*"] \\
{[Z,X]} \arrow[r, "g_*"] & {[Z,Y]}.
\end{tikzcd}$$

Show that (1) they are commutative, and (2) the vertical maps are all isomorphisms.

Next we show that every map is homotopy equivalent to a cofibration.

Problem 5.20 Let $f : A \to X$ be any map, and let $M_f = X \cup_f (A \rtimes I)$; in diagrams,[5] M_f is the pushout in the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d] \\
A \rtimes I \arrow[r] & M_f.
\end{tikzcd}$$

The space M_f is called the (pointed) **mapping cylinder** of f.

(a) Show that there is a pushout square

$$\begin{tikzcd}
A \vee A \arrow[r, "f \vee \mathrm{id}_A"] \arrow[d, "(i_0,i_1)"'] \arrow[dr, phantom, "\text{pushout}"] & X \vee A \arrow[d] \\
A \rtimes I \arrow[r] & M_f.
\end{tikzcd}$$

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[4] This definition makes sense in either the pointed or the unpointed context – we will only use it in the pointed context.

[5] and more precisely!

(b) Conclude that the maps $j : A \to M_f$ given by $j(a) = (a, 1)$ and $r : X \to M_f$ given by $x \mapsto (x, 0)$ are cofibrations.

(c) Show that M_f and X are homotopy equivalent.

HINT Find a map $r : X \to M_f$ so that $X \hookrightarrow M_f \to X$ is the identity map.

(d) Show that the diagram

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, equal] & M_f \arrow[d, "\simeq"] \\
A \arrow[r, "f"] & X
\end{tikzcd}$$

is strictly commutative, and conclude that f is homotopy equivalent to a cofibration.

(e) Show that if $f : A \to X$ is a map in $\mathcal{T}_*$, then the whole diagram can be constructed in $\mathcal{T}_*$.

(f) Show that this whole discussion is natural. Carefully state what categories and functors are involved.

You have proved the following theorem.

Theorem 46 *Every map $f : A \to X$ in $\mathcal{T}_*$ fits into a strictly commutative diagram*

$$\begin{tikzcd}
A \arrow[r, "j"] \arrow[d, equal] & M_f \arrow[d, "q"] \\
A \arrow[r, "f"] & X
\end{tikzcd}$$

in which j is a cofibration and q is a homotopy equivalence in $\mathcal{T}_*$ with homotopy inverse r. Furthermore, the entire construction is functorial on the category of maps in $\mathcal{T}_*$.

The process of replacing a nasty map f with a homotopy equivalent cofibration is often referred to as 'converting f to a cofibration.'

Definition 47 The (standard) **cofiber** of a map $f : A \to X$ is the map $X \to C_f$ obtained as a result of the following procedure:

STEP 1 convert f to the cofibration $j : A \to M_f$, and

STEP 2 take the composite of $X \hookrightarrow M_f$ with the quotient map $M_f \to M_f/A$ to obtain $X \to C_f$.

The space C_f and is also called the **mapping cone** of f.

Problem 5.21 Show that there is a pushout square

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d] \\ CA \arrow[r] & C_f, \end{tikzcd}$$

where $i : A \hookrightarrow CA$ is the standard inclusion.

Note that the cofiber is not just the space C_f, it is the map $X \to C_f$. However, the usual practice is to be sloppy and refer to C_f as the cofiber.

It is possible to convert a map to a cofibration in many different ways,[6] though the mapping cylinder construction is the standard method. No matter what construction is used, the resulting quotient space is referred to as a cofiber for the given map. It is natural to ask how different the corresponding cofibers can be. We will show in Problem 6.43 that if C and C' are cofibers arising from two different constructions, then they are homotopy equivalent to each other.

Problem 5.22 What is the cofiber of the unique map $S^1 \to *$? Describe the cofiber of $X \to *$ in general.

Given any map $f : A \to X$ we get a cofiber $X \to C_f$; together these form a sequence of maps $A \to X \to C_f$. Any such sequence, or any sequence that is naively homotopy equivalent to such a sequence, is called a **cofiber sequence**. We conclude this section by showing that cofiber sequences, used in the domain, yield exact sequences of homotopy sets.

Problem 5.23 Let Y be any space, and let $A \xrightarrow{f} B \xrightarrow{g} C$ be a cofiber sequence. Show that the sequence

$$\begin{tikzcd} {[A, Y]} & {[B, Y]} \arrow[l, "f^*"'] & {[C, Y]} \arrow[l, "g^*"'] \end{tikzcd}$$

is an exact sequence of pointed sets.

Unpointed Mapping Cylinders and Mapping Cones. If we are working with unpointed spaces, then our homotopies have to be free homotopies. This forces us to define the unpointed mapping cylinder using $A \times I$ instead

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[6] For example, if the map is a cofibration already, why do anything to it?

of $A \rtimes I$, using the pushout square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d] \\
A \times I \arrow[r] & M_f.
\end{tikzcd}$$

The unpointed mapping cone is obtained from the unreduced mapping cylinder by collapsing M_f to a point.

Problem 5.24 Let X be a cofibrant space. Then after forgetting the basepoints and attaching disjoint ones, the inclusion $* \to X$ of the basepoint becomes the pointed map $*_+ \to X_+$.

(a) Show that the cofiber of this map is X itself.

(b) Show that there is an exact sequence

$$\pi_0(Y) \leftarrow \langle X, Y\rangle \leftarrow [X, Y]$$

(c) Conclude that if Y is path-connected, then the map $[X, Y] \to \langle X, Y\rangle$ is surjective. Is it is bijective?

5.5 Fibrations

Fibrations are the dual concept to cofibrations, and our discussion of them is largely parallel to, but briefer than, our discussion of cofibrations.

Dualizing Cofibrations. To see the duality clearly, we rewrite the homotopy extension property using the cylinder object notation, like so

$$\begin{tikzcd}
A \arrow[r, "i"] \arrow[d, "i_0"'] & X \arrow[d, "i_0"] \arrow[ddr, bend left, "f"] & \\
\mathrm{Cyl}(A) \arrow[r, "\mathrm{Cyl}(i)"] \arrow[drr, bend right, "H_A"'] & \mathrm{Cyl}(X) \arrow[dr, dotted, "H"] & \\
 & & Y.
\end{tikzcd}$$

Since dualization reverses arrows and replaces cylinder objects with path objects, we find that a map $p : E \to B$ satisfies the dual property – called the **homotopy lifting property** – if it satisfies the condition set out by the diagram

$$\begin{tikzcd}
X \arrow[rrd, bend left, "K_B"] \arrow[rd, dotted, "K"] \arrow[rdd, bend right, "f"'] & & \\
& \mathrm{Path}(E) \arrow[r] \arrow[d, "p_0"] & \mathrm{Path}(B) \arrow[d, "p_0"] \\
& E \arrow[r, "p"] & B.
\end{tikzcd}$$

If the map p has this property for every map f it is called a **fibration**.

This is a fine definition, but what does it mean? We are given a map $f : X \to E$, and hence the composite map $\phi = p \circ f : X \to B$; we are also given a homotopy K_B from ϕ to some other map γ. If p is a fibration, then a new homotopy K from f to some other map g such that $p \circ g = \gamma$ can be found. This is frequently expressed in the diagram form

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] & E \arrow[d, "p"] \\
\mathrm{Cyl}(X) \arrow[r, "H_B"] \arrow[ru, dotted, "H"] & B.
\end{tikzcd}$$

The idea here is that you have a *partial lift* of H – that is, the map f lifts the part of H along the bottom level of the cylinder. The question is whether or not that partial lift can be extended to a full lift. This is why the defining property of fibrations is called the **homotopy lifting property**.

PROBLEM 5.25 Formulate and prove the dual of Problem 5.6. You will find that $p : E \to B$ is a fibration if and only if there is a map from a certain pullback to E^I with certain properties. Such a map is called a **lifting function**; lifting functions play a key role in in the detailed point-set level study of fibrations.

PROBLEM 5.26

(a) Show that the map $E \to *$ is a fibration no matter what E is. Thus every space E is **fibrant**.

(b) Prove that the evaluation map $p_0 : B^I \to B$ is a fibration, no matter what A is.[7]

(c) Show that the projection $\mathrm{pr}_B : A \times B \to B$ is a fibration for any A and B.

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[7] Likewise, p_1 is a fibration.

The following relative homotopy lifting property is frequently useful.

Problem 5.27 Suppose $i : A \hookrightarrow X$ is a cofibration and $p : E \to B$ is a fibration. Suppose we are given a map $f : X \to B$, a homotopy $H : X \times I \to B$, a map $\phi : X \to E$ such that $p \circ \phi = f$, and a homotopy $J : \phi|_A \simeq \gamma_A$ such that $p \circ J = H|_A$.

(a) Draw a diagram illustrating all this information. It will behoove you to introduce the space $T = X \cup (A \times I)$.

(b) Show that there is a homotopy $\widetilde{H} : \phi \simeq \gamma$ such that $p \circ \widetilde{H} = H$ and $\widetilde{H}|_{A \times I} = J$.

Hint Use Proposition 39.

Theorem 48 *Suppose the square*

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\
A \arrow[r] & B
\end{tikzcd}$$

is a pullback square, and that p is a fibration. Then q is also a fibration.

Problem 5.28 Prove Theorem 48 by studying the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "i_0"'] & E_A \arrow[r] \arrow[d, "p_A"] & E \arrow[d, "p"] \\
X \times I \arrow[r] \arrow[ru, dotted] \arrow[rru, dotted] & A \arrow[r] & B.
\end{tikzcd}$$

When $p : E \to B$ is a fibration, and $b \in B$, we call the space $p^{-1}(b) \subseteq E$ the **fiber of p over b**.[8] Two different methods of converting f to a fibration will yield different homotopy fibers; however we will show in Problem 6.43 that any two homotopy fibers for a given map f will be homotopy equivalent. When B is a pointed space, we call $p^{-1}(*)$ the **fiber of p**.

Exercise 5.29 Describe F as a pullback. Is F a pointed space? What is the basepoint? What if p is a map in $\mathcal{T}_*$?

Just as for cofibrations, we can convert any map $f : X \to Y$ into a fibration. Here's how to do it. Let E_f be the pullback in the diagram

$$\begin{tikzcd}
E_f \arrow[r, "g"] \arrow[d, "q"'] \arrow[rd, phantom, "\text{pullback}"] & Y^I \arrow[d, "p_0"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

---

[8] It can (and will) be shown that if B is path connected then any two fibers are homotopy equivalent to each other.

Explicitly,

$$E_f = \{(x, \alpha) \in X \times Y^I \mid \alpha(0) = f(x)\},$$

and q and g are the projections on the first and second coordinate, respectively. We define a map $p : E_f \to Y$ by the formula $p(x, \alpha) = \alpha(1)$.

Problem 5.30

(a) Show that $q : E_f \to X$ is a homotopy equivalence whose inverse $r : X \to E_f$ is given by $x \mapsto (x, \tau_{f(x)})$, where $\tau_{f(x)}$ is the constant path at $f(x)$.

Hint Gently shrink the path coordinate of E_f down to a constant path.

(b) Show that $p : E_f \to Y$ is a fibration.

(c) Show that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & Y \arrow[d, equal] \\ E_f \arrow[r, "p"] & Y \end{tikzcd}$$

is strictly commutative and conclude that f is homotopy equivalent to a fibration.

(d) Show that the whole discussion is natural, including the maps q and r.

Theorem 49 *Every map $f : A \to X$ in $\mathcal{T}_*$ fits into a strictly commutative diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"', "\simeq"] & Y \arrow[d, equal] \\ E_f \arrow[r, "p"] & Y \end{tikzcd}$$

in which p is a fibration and r is a homotopy equivalence in $\mathcal{T}_*$. Furthermore, the entire construction is functorial on the category of maps in $\mathcal{T}_*$.

The process of replacing a map with a homotopy equivalent fibration is referred to a ‘converting f to a fibration.’

Definition 50 The (standard) **homotopy fiber** of a map $f : X \to Y$ in $\mathcal{T}_*$ is the map $i : F \to X$ obtained by

Step 1 converting f to the fibration $p : E_f \to Y$, and

Step 2 setting $F = p^{-1}(*)$ and i to be the composition $F \to E_f \to X$.

Any sequence of the form $F \to X \to Y$, where $F \to X$ is the homotopy fiber of $X \to Y$, or any sequence homotopy equivalent to such a sequence, is called a **fibration sequence** or a **fiber sequence**.

PROBLEM 5.31 Determine the homotopy fiber of the unique map $* \to X$.

PROBLEM 5.32 Let $p : E \to B$ be a map in $\mathcal{T}_*$.

(a) Suppose p is a fibration, let $F = p^{-1}(*) \subseteq E$ be the fiber of p and write $i : F \to E$ for the inclusion map. Show that for any space X, the sequence of pointed sets

$$[X, F] \xrightarrow{i_*} [X, E] \xrightarrow{p_*} [X, B]$$

is exact.

(b) Suppose $F \xrightarrow{i} X \xrightarrow{p} Y$ is a fibration sequence in $\mathcal{T}_*$. Show that the induced sequence of pointed sets

$$[X, F] \xrightarrow{i_*} [X, E] \xrightarrow{p_*} [X, B]$$

is exact for every $X \in \mathcal{T}_*$.

Pointed Fibrations. We have chosen to study unpointed fibrations in detail; but we should also mention the pointed fibrations. The pointed version requires that in any diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \rtimes I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & B. \end{tikzcd}$$

the map $\widetilde{H}$ can be filled in to make the diagram commute.

It is not true that every unpointed fibration is also a pointed fibration. However, this is *almost* true, as you will show in the next problem.

PROBLEM 5.33 Let $p : E \to B$ be a map in $\mathcal{T}_*$.

(a) Show that $p : E \to B$ is a pointed fibration if and only if in any diagram in $\mathcal{T}_*$ of the form

$$\begin{tikzcd} X \cup * \times I \arrow[r, "\bar{f}"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ X \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & B. \end{tikzcd}$$

where $\bar{f}(* \times I) = *$, the map $\widetilde{H}$ can be filled in so that the whole diagram commutes.

(b) Let $p : E \to B$ be an unpointed fibration. Show that if X is cofibrant, then The map $\widetilde{H}$ can be filled in.

Since we will deal almost exclusively with cofibrant spaces, there in practice very little differentce between pointed and unpointed fibrations.

5.6 Fibrations, Cofibrations and Mapping Spaces

We end this section by studying the effect of mapping space functors on fibration and cofiberation sequences. For any $f : E \to B$ and any space X, we get an induced map

$$f_* : \mathrm{map}(X, E) \to \mathrm{map}(X, B).$$

On the other hand, if $g : A \to X$ and Y is any space, then we get an induced map

$$g^* : \mathrm{map}(X, Y) \to \mathrm{map}(A, Y).$$

Our theorem tells us what happens when f is a fibration or if g is a cofibration.

Theorem 51

(a) If $f : A \to X$ is a cofibration, then for any space Y, the induced map $f^* : \mathrm{map}(X, Y) \to \mathrm{map}(A, Y)$ is a fibration.

(b) If $p : E \to B$ is a fibration, then for any space X, the induced map $p_* : \mathrm{map}(X, E) \to \mathrm{map}(X, B)$ is a fibration.

EXERCISE 5.34 There is a curious thing happening here: in both the statements, the conclusion is that the induced map is a *fibration* – neither is a cofibration! Is this a failure of duality?

PROBLEM 5.35

(a) Write down the diagram that must be filled in in order for g^* to be a fibration – that is, the diagram that defines the homotopy lifting property for maps out of W.

(b) Your diagram involves three 'known' maps from W into various mapping spaces. Use the exponential law to rewrite those maps as maps involving products instead. Make note of all the relationships between these maps that are imposed from the original diagram.

(c) Your goal is to produce a map from W to a certain mapping space. Rewrite this hypothetical map in terms of products rather than mapping spaces, and make note of its relationships to the maps from part (b).

(d) Assemble your work from parts (b) and (c) in a single diagram with a dotted arrow. Explain why that dotted arrow can be filled in.

(e) What is the fiber of g^*?

You have proven Theorem 51(a).

PROBLEM 5.36 Prove Theorem 51(b) and determine the fiber of the fibration.

HINT Set up the diagrams describing the required maps, and use the exponential law to redraw them.

Theorem 52

(a) *If $A \to B \to C$ is a cofibration sequence then*

$$\mathrm{map}(A,X) \leftarrow \mathrm{map}(B,X) \leftarrow \mathrm{map}(C,X)$$

is a fibration sequence

(b) *$X \to Y \to Z$ is a fibration sequence then*

$$\mathrm{map}(A,X) \to \mathrm{map}(A,Y) \to \mathrm{map}(A,Z)$$

is a fibration sequence.

PROBLEM 5.37 Prove Theorem 52.

We have stated this theorem for unpointed mapping spaces, but it is also true for pointed mapping spaces.

PROBLEM 5.38 Let $X, Y \in \mathcal{T}_*$, and denote the basepoints by $x_0 \in X$ and $y_0 \in Y$.

(a) Show that there is a pullback diagram

$$\begin{tikzcd}
\mathrm{map}_*(X,Y) \arrow[r] \arrow[d] & \mathrm{map}(X,Y) \arrow[d, "@_{x_0}"] \\
\{y_0\} \arrow[r] & Y.
\end{tikzcd}$$

(b) Prove the pointed analog of Theorem 51.

Theorem 51 and its pointed analog have many crucial applications.

PROBLEM 5.39 Let $x \in X$, and let $i : \{x\} \hookrightarrow X$ be the inclusion map.

(a) Using the identification $\mathrm{map}(\{x\}, Y) \cong Y$ of Exercise 3.13, show that $i^* = @_x : \mathrm{map}(X,Y) \to \mathrm{map}(\{x\}, Y)$.

(b) Show that if the inclusion $i : \{x\} \hookrightarrow X$ is a cofibration, then $@_x$ is a fibration.

When we use the unit interval I in the pointed homotopy category $\mathcal{T}_*$, we use 0 as the basepoint. The space of pointed maps from I to X

$$\mathcal{P}(X) = \mathrm{map}_*(I, X)$$

is extremely important in homotopy theory. It is called the **path space** of X.

Problem 5.40

(a) Show that the map $X^I \to X \times X$ given by $f \mapsto (f(0), f(1))$ is a fibration.

(b) Show that the map $\mathcal{P}(X) \to X$ given by $f \mapsto f(1)$ is a fibration.

(c) Reprove Problem 5.26(b).

Determine the fibers.

Chapter 6

Homotopy Limits and Colimits

If $F, G : \mathcal{I} \to \mathcal{C}$ are two diagrams, and $F \to G$ is a pointwise equivalence, then the induced map of colimits is an equivalence. But what if $F, G : \mathcal{I} \to \mathcal{T}_*$ and the diagram map $F \to G$ is only a pointwise *homotopy* equivalence? Unfortunately, examples show that we cannot conclude the induced map is a homotopy equivalence without some extra hypotheses on the diagrams F and G. If the diagrams F and G are sufficiently nice, then a pointwise homotopy equivalence $F \to G$ *does* induce a homotopy equivalence of colimits. It turns out that every diagram is pointwise equivalent to a nice one, and we define the homotopy colimit of F to be the colimit of a pointwise equivalent nice diagram $\overline{F}$. This is well defined up to homotopy equivalence; and it can be made functorial (with a functorial choice of nice approximation).

We will concentrate on homotopy colimits (and limits) in the category of *pointed* spaces $\mathcal{T}_*$. The development of the theory for unpointed spaces is entirely parallel. Unpointed homotopy colimits and their relation to the pointed homotopy colimits will be discussed in some detail in the final section of the chapter.

6.1 Homotopy Equivalence in Diagram Categories

Let $\mathcal{I}$ be a small category, and consider category $\mathcal{T}^{\mathcal{I}}$ of $\mathcal{I}$-shaped diagrams in $\mathcal{T}$. What exactly should it mean when we say two such diagrams are homotopy equivalent? There are at least two reasonable definitions, and the interplay between them is of crucial importance to our work.

Pointwise Homotopy Equivalence. The simplest idea is what we will call **pointwise homotopy equivalence** of diagrams. We say that a map of diagrams $\phi : F \to G$ is a pointwise homotopy equivalence if for each object $i \in \mathcal{I}$, the map $\phi(i) : F(i) \to G(i)$ is a homotopy equivalence in $\mathcal{T}_*$ (thus the 'points' referred to in the word 'pointwise' are the objects of the indexing category $\mathcal{I}$). If ϕ is a pointwise homotopy equivalence, then each $\phi(i)$ has a homotopy inverse in $\mathcal{T}_*$, but no claims are made about how those homotopy inverses are related to one another, or about the homotopies that are implicit here.

Diagram Homotopy. To get some control over these homotopy inverses and homotopies, we need to talk about homotopies of maps of diagrams. If $F \in \mathcal{T}_*^{\mathcal{I}}$ and X is a space, then we can define $F \rtimes X$ to be the functor defined on objects $i \in \mathcal{I}$ and morphisms $\alpha : i \to j$ in $\mathcal{I}$ by

$$F \rtimes X(i) = F(i) \rtimes X \quad \text{and} \quad F \rtimes X(\alpha) = F(\alpha) \rtimes \mathrm{id}_X.$$

This construction is clearly natural in both variables.

Exercise 6.1 What exactly does it mean for the construction to be 'natural in both variables?'

Exercise 6.2 Show that $\operatorname{colim}(F \rtimes X) \cong (\operatorname{colim} F) \rtimes X$.

Naturality implies that the functor $F \times I$ comes with two 'inclusion functors'

$$\mathrm{in}_0 : F \to F \rtimes I \qquad \text{and} \qquad \mathrm{in}_1 : F \to F \rtimes I.$$

Now we can define a (pointed) **homotopy** between two diagram maps (i.e., natural transformations) in the obvious way: a homotopy between $\phi_0, \phi_1 : F \to G$ is a natural transformation

$$H : F \rtimes I \to G$$

such that the diagram of natural transformations

$$\begin{tikzcd}
F \arrow[r, "\mathrm{in}_0"] \arrow[dr, bend right, "\phi_0"'] & F \rtimes I \arrow[d, "H"] & F \arrow[l, "\mathrm{in}_1"'] \arrow[dl, bend left, "\phi_1"] \\
& G &
\end{tikzcd}$$

commutes.

In terms of spaces, a diagram homotopy is a big collection of homotopies, one for each object $i \in I$, and they are required to be compatible with one another in the sense that for each $\alpha : i \to j$ in $\mathcal{I}$, the diagram

$$\begin{tikzcd}
F(i) \rtimes I \arrow[r, "F(\alpha)\rtimes \mathrm{id}_I"] \arrow[d, "H(i)"'] & F(j) \rtimes I \arrow[d, "H(j)"] \\
G(i) \arrow[r, "G(\alpha)"] & G(j)
\end{tikzcd}$$

is commutative.

Here is a nice property of diagram homotopy.

Proposition 53 *Let $\phi_0, \phi_1 : F \to G$, and let $H : \phi_0 \simeq \phi_1$. Then the induced maps $f_0, f_1 : \operatorname{colim} F \to \operatorname{colim} G$ are (pointed) homotopic.*

Problem 6.3 Prove Proposition 53

Hint The map of diagrams H induces a map J of colimits.

Diagram Homotopy Equivalence. Now that we have a definition of homotopy, we can define homotopy equivalence: we call a map of diagrams $\phi : F \to G$ a **diagram homotopy equivalence** if there is a $\theta : G \to F$ such that $\theta \circ \phi \simeq \mathrm{id}_F$ and $\phi \circ \theta \simeq \mathrm{id}_G$.

Exercise 6.4 Show that every diagram homotopy equivalence is a pointwise homotopy equivalence.

Now we have the following instant corollary of Proposition 53.

Corollary 54 *If $\phi : F \to G$ is a diagram homotopy equivalence, then the induced map $f : \operatorname{colim} F \to \operatorname{colim} G$ is a homotopy equivalence in $\mathcal{T}$.*

Exercise 6.5 Prove it!

Here's what we have done so far: we have given two (apparently different) notions of homotopy equivalence of diagrams. The first one, *pointwise homotopy equivalence*, is conceptually simple, and the second one, *diagram homotopy equivalence*, has the extremely nice property that a diagram homotopy equivalence induces a homotopy equivalence of colimits. This leads us to a crucial question: *is every pointwise homotopy equivalence $\phi : F \to G$ automatically a homotopy equivalence of diagrams?*

Exercise 6.6 Find an example of a pointwise homotopy equivalence that is not a diagram homotopy equivalence.

Hint What is the pushout of $CX \leftarrow X \to CX$?.

Pointwise Equivalence in h$\mathcal{T}$. There is yet another way to apply the idea of homotopy equivalence to diagrams. Let $L : \mathcal{T}_* \to \text{H}\mathcal{T}_*$ be the functor from the topological category to the homotopy category defined in Section 4.3. It may be that we have two diagrams $F, G : \mathcal{I} \to \mathcal{T}_*$ that are connected by maps that result in a diagram that commutes only up to homotopy. While these maps do not define a diagram map in $\mathcal{T}_*$, when we apply L, we do have a diagram map $L \circ F \to L \circ G$. If this map is a pointwise equivalence – i.e., if all the maps involved in the original homotopy commutative diagram joining F to G are homotopy equivalences – then we say that F and G are *pointwise equivalent in the homotopy category* $\text{H}\mathcal{T}_*$. In the special case that the diagram is a single map, this is just homotopy equivalence of maps as defined in Definition 45. This is a much weaker notion of equivalence for diagrams than the ones defined above, and we will not use it at all in this chapter. In the next chapter and later chapters it will take on much greater significance.

6.2 Homotopy Colimits of Diagrams

In this section we define homotopy colimits. The idea is to replace the given diagram with a 'nice' diagram that is equivalent, and form the categorical colimit of the replacement.

6.2.1 Left Nice Diagrams

To be precise about the word *nice*, we first have to talk about pointwise fibrations. We call a diagram map $\phi : F \to G$ in $\mathcal{T}_*^{\mathcal{I}}$ a **pointwise fibration** if for each $i \in \mathcal{I}$ the map $\phi(i) : F(i) \to G(i)$ is a fibration in $\mathcal{T}_*$.

Problem 6.7 Let $\phi : F \to G$ be a map of diagrams. Show that there is a factorization

$$\begin{tikzcd}
F \arrow[rr, "\phi"] \arrow[dr, "e"'] & & G \\
& \widetilde{F} \arrow[ur, "f"'] &
\end{tikzcd}$$

of ϕ such that

1. e is a diagram homotopy equivalence
2. f is a pointwise fibration
3. there is a map $\overline{e} : \widetilde{F} \to F$ such that $\overline{e} \circ e = \text{id}_F$.
4. all this structure is functorial.

Hint Use Theorem 49.

Definition 55 A diagram N is **left nice** if for every diagram

$$\begin{tikzcd} & F \arrow[d, "e"] \\ N \arrow[r, "f"] \arrow[ur, dotted, bend left] & G, \end{tikzcd}$$

of functors $\mathcal{I} \to \mathcal{T}_*$, where e is a pointwise homotopy equivalence and a pointwise fibration, the dotted arrow can be filled in to make the triangle commute.

Problem 6.8 Show that it is equivalent to drop the fibration condition on e and ask only for a homotopy commutative triangle.

Here's a key property of left nice diagrams.

Proposition 56 *If F and G are left nice, and $\phi : F \to G$ is a pointwise homotopy equivalence, then ϕ is a diagram homotopy equivalence.*

Problem 6.9 Prove it!

The main theorem of this section asserts that for every diagram $F : \mathcal{I} \to \mathcal{T}$, there is a left nice diagram $N : \mathcal{I} \to \mathcal{T}_*$ and a pointwise equivalence $N \to F$. This equivalence is called a **left nice approximation** of F. It is not unique, but it can be done functorially.

Theorem 57 *For each diagram F, there is a pointwise equivalence $c : \overline{F} \to F$, where $\overline{F}$ is left nice; in fact, the construction of $\overline{F}$ can be done functorially.*

We will take the general statement for granted. Specific constructions for the cases we will use most frequently will be given in detail in Section 6.4.

Though left nice replacement can be done functorially, it is often convenient to use some other *ad hoc* replacement. For example, if we recognize that a given diagram is already nice, why modify it at all?

Adjoint Functors and Nice Diagrams. It is useful to be able to recognize left and right nice diagrams, so we ask the question: if $N : \mathcal{I} \to \mathcal{T}_*$ is a nice diagram, and $F : \mathcal{T}_* \to \mathcal{T}_*$ is another functor, how can we tell whether or not $F \circ N : \mathcal{I} \to \mathcal{T}_*$ is nice?

Theorem 58 *Let L, R be an adjoint pair of functors $\mathcal{T}_* \to \mathcal{T}_*$. If R preserves fibrations and homotopy equivalences, then if N is a left nice diagram, so is $L \circ N$.*

PROBLEM 6.10 Let L and R be an adjoint pair.

(a) Show that the lifting problems

$$\begin{tikzcd} & A \arrow[d, "f"] \\ LX \arrow[r, "g"] \arrow[ur, dotted, bend left, "\lambda"] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & RA \arrow[d, "R(f)"] \\ Y \arrow[r, "\widetilde{g}"] \arrow[ur, dotted, bend left, "\widetilde{\lambda}"] & RB \end{tikzcd}$$

are equivalent in the sense that λ exists if and only if $\widetilde{\lambda}$ exists.

(b) Prove Theorem 58.

PROBLEM 6.11 If F is left nice, then so are $F \times A$, $F \rtimes A$ and $F \wedge A$.

6.2.2 Homotopy Colimits

We are interested in homotopy theory, and we would like to use colimits to build new spaces out of old ones. This leads to the question 'does every diagram in the homotopy category $\mathrm{H}\mathcal{T}_*$ have a colimit?' We will find examples of prepushout diagrams in $\mathrm{H}\mathcal{T}_*$ that have no pushout, and similarly for pullbacks, so the answer to this question is 'no.' The homotopy colimit construction is the 'best possible' homotopy invariant approximation to the colimit construction.

Definition 59 A space X is a **homotopy colimit** for a diagram $F : \mathcal{I} \to \mathcal{T}_*$ if $X \simeq \operatorname{colim} \overline{F}$, where $\overline{F} \to F$ is a left nice replacement of F.

Because of Theorem 57, each diagram has a left nice replacement, and hence a homotopy colimit.

PROBLEM 6.12 Recall that we have used the notation $L : \mathcal{T}_* \to \mathrm{H}\mathcal{T}_*$ for the functor from the category of pointed spaces to the corresponding homotopy category. Let $F : \mathcal{I} \to \mathcal{T}_*$ be a diagram, and let X be a homotopy colimit. Being a topological space, X can be considered as an object of $\mathcal{T}_*$ or of $\mathrm{H}\mathcal{T}_*$.

(a) Show that there is a diagram map (in the homotopy category $\mathrm{H}\mathcal{T}_*$) $j : L \circ F \to \Delta_X$ with the following nice property:

($\star$) if $Y \in \mathrm{H}\mathcal{T}_*$ and there is a diagram map $g : \mathcal{L} \circ F \to \Delta_Y$, then there is a map $f : X \to Y$ such that

$$\begin{tikzcd} L \circ F \arrow[rr] \arrow[dr] & & \Delta_Y \\ & \Delta_X \arrow[ur, dotted, "\Delta_f"'] & \end{tikzcd}$$

commutes in the category $(\mathrm{H}\mathcal{T}_*)^{\mathcal{I}}$.

(b) Show that if $L \circ F$ has a colimit Q, then Q is a retract (in $\mathrm{H}\mathcal{T}_*$) of X.

The space X would be the categorical pushout if the map f were unique.

Problem 6.13 Show by example that the map Δ_f need not be unique.

It is important to observe that the left nice replacement $\overline{F}$ is not unique, and so the space $\operatorname{hocolim} F$ is not uniquely determined by the definition; rather, the definition identifies a long list of spaces which qualify as homotopy colimits of F.

6.3 Induced Maps of Homotopy Colimits

If $F \to G$ is a map of diagrams, then the formal properties of colimits yield a unique map $\operatorname{colim} F \to \operatorname{colim} G$. By contrast, the homotopy colimit of F is not (usually) the categorical colimit of F, and so a map of diagrams does not give rise to a map of homotopy colimits in the same purely formal way. Let us investigate what we can obtain from a map of diagrams.

Problem 6.14 Let $\phi : F \to G$ be a diagram map, and let $\overline{F} \to F$ and $\overline{G} \to G$ be left nice replacements.

(a) Show that there is a map $\overline{\phi} : \overline{F} \to \overline{G}$ which is compatible with ϕ in the sense that the square

$$\begin{tikzcd}
\overline{F} \arrow[r, dotted, "\overline{\phi}"] \arrow[d] & \overline{G} \arrow[d] \\
F \arrow[r, "\phi"] & G
\end{tikzcd}$$

commutes up to diagram homotopy. Could there be more than one choice for $\overline{\phi}$?

(b) Show that if ϕ is a pointwise homotopy equivalence, then $\overline{\phi}$ is a diagram homotopy equivalence.

(c) Show that if $\overline{F}$ and $\widetilde{F}$ are two nice replacements for F, then there is a diagram homotopy equivalence $\overline{F} \simeq \widetilde{F}$.

Problem 6.14 implies a kind of limited naturality to the construction.

Problem 6.15 Let $\phi : F \to G$ be a map of diagrams; let X be a homotopy colimit for F, and let Y be a homotopy colimit for G. Show that ϕ induces a map $\Phi : X \to Y$.

The map Φ is not uniquely determined by ϕ, even up to homotopy. We should think of a map of diagrams ϕ as inducing a *set* of maps between homotopy colimits rather than a single map. However these maps are strongly related to one another.

Problem 6.16 Let $\phi : F \to G$ be a diagram map. Choose left nice approximations

$$\overline{F} \to F, \qquad \widetilde{F} \to F, \qquad \overline{G} \to G, \qquad \text{and} \qquad \widetilde{G} \to G.$$

(a) Show that there is a diagram of diagram maps

$$\begin{tikzcd}
\widetilde{F} \arrow[rrr, "\widetilde{\phi}"] \arrow[dd, "\simeq"'] \arrow[dr] & & & \widetilde{G} \arrow[dd, "\simeq"] \arrow[dl] \\
& F \arrow[r, "\phi"] & G & \\
\overline{F} \arrow[rrr, "\overline{\phi}"] \arrow[ur] & & & \overline{G}. \arrow[ul]
\end{tikzcd}$$

which commutes up to diagram homotopy.

(b) Show that any two induced maps of homotopy colimits are homotopy equivalent maps.

(c) Conclude that any two induced maps have the same connectivity. In particular, show that if one induced map is a (weak) homotopy equivalence, then they all are.

(d) Show that if X and Y are homotopy colimits for F, then $X \simeq Y$.

Now that we know that the homotopy colimit of a diagram is well defined up to homotopy type, we can speak of *the* homotopy colimit of F. We will sometimes denote it by $\operatorname{hocolim} F$.

Exercise 6.17

(a) Find all the maps $S^1 \to S^1$ induced by the diagram map

$$\begin{tikzcd}
* \arrow[d] & S^0 \arrow[l] \arrow[r] \arrow[d, "\mathrm{id}"] & * \arrow[d] \\
* & S^0 \arrow[l] \arrow[r] & *.
\end{tikzcd}$$

(b) Find all the maps induced by the diagram

$$\begin{tikzcd}
* \arrow[d] & S^0 \arrow[l] \arrow[r] \arrow[d] & * \arrow[d] \\
S^1 & * \arrow[l] \arrow[r] & S^1.
\end{tikzcd}$$

Induced Maps Between Suspensions. We know that the homotopy pushout of $* \leftarrow X \rightarrow *$ is the suspension ΣX. Now let $f : X \rightarrow Y$ and consider the map of prepushout diagrams

$$\begin{tikzcd}
* \arrow[d] & X \arrow[l] \arrow[r] \arrow[d, "f"] & * \arrow[d] \\
* & Y \arrow[l] \arrow[r] & *.
\end{tikzcd}$$

This map induces maps $\Sigma X \rightarrow \Sigma Y$ of the homotopy pushouts, but which maps are induced and which are not? To be able to compare maps to one another, we choose the standard suspensions as our homotopy colimits, so that each of them is a union of two cones:

$$\Sigma X = C_+X \cup C_-X \qquad \text{and} \qquad \Sigma Y = C_+Y \cup C_-Y.$$

With these choices, we can see that the nice replacements of the given given diagram include

$$\begin{tikzcd}
C_-X \arrow[d, "Cf"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & C_+X \arrow[d, "Cf"] \\
C_-Y & Y \arrow[l] \arrow[r] & C_+Y
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
C_+X \arrow[d, "Cf"'] & X \arrow[l] \arrow[r] \arrow[d, "f"] & C_-X \arrow[d, "Cf"] \\
C_-Y & Y \arrow[l] \arrow[r] & C_+Y.
\end{tikzcd}$$

The map induced by the first replacement is Σf, but the second diagram induces $-\Sigma f$. Thus, as long as there is a map $f : X \rightarrow Y$ whose suspension Σf does not have order 2, we will have an example where more than one homotopy class of maps is induced by a given map of prepushout diagrams.

EXERCISE 6.18 Is it possible that the given diagram induces other maps besides $\pm\Sigma f$?

Functorial Homotopy Colimits. Sometimes the freedom of using any nice replacement we like is outweighed by the desirability of having well-defined and natural induced maps. If this is the case, we can fall back on a specific functorial construction (which exists by Theorem 57). Let's choose such a functor, and denote it by

$$\mathrm{Nice}_L : \mathcal{T}_*^{\mathcal{I}} \rightarrow \mathcal{T}_*^{\mathcal{I}},$$

which comes with a natural transformation $\mathrm{Nice}_L \rightarrow \mathrm{id}$, so that for any $F \in \mathcal{T}^{\mathcal{I}}$, the map $\mathrm{Nice}_L(F) \rightarrow F$ is a left nice replacement. Now we can define a homotopy colimit *functor* by setting

$$\mathrm{hocolim} : \mathcal{T}_*^{\mathcal{I}} \rightarrow \mathcal{T}_* \qquad \text{by the rule} \qquad \mathrm{hocolim}\, F = \mathrm{colim}\, \mathrm{Nice}_L(F).$$

and for $\phi : F \to G$, $\operatorname{colim}(\mathrm{Nice}_L(\phi))$ is the induced map of homotopy colimits.

6.4 Special Cases: Pushouts, 3×3s and Telescopes

In this section we give the explicit constructions for three important diagram shapes. The proofs that they actually do the job are based on a single quite technical result about cofibrations. The proof of this result is quite involved, and presenting it here would interrupt the progression of ideas, so the proof is (or will be) given in Appendix A.

6.4.1 Nice Maps

The simplest kind of diagram, other than a single space, is just a single map $X \to Y$. We will show that every cofibration is a left nice diagram. But even more is true: we get to choose some (but not all) of the maps and homotopies in the lifting problem that defines left nice maps. The 'supernice' property of cofibrations is the key to our constructions of nice replacements for all other diagrams.

Let us consider the lifting problem

$$\begin{tikzcd}
 & \boxed{A \to B} \arrow[d, "e"] \\
\boxed{X \to Y} \arrow[ur, dotted, "\lambda"] \arrow[r, "f"] & \boxed{P \to Q}
\end{tikzcd}$$

in the category $(\mathcal{T}_*)^{\bullet \to \star}$ of single-map diagrams. If the map $\lambda = (\lambda_X, \lambda_Y)$ exists, then we have two homotopy commutative triangles

$$\begin{tikzcd}
 & A \arrow[d, "e_A"] & & & B \arrow[d, "e_B"] \\
X \arrow[ur, bend left, "\lambda_X"] \arrow[r, "f_X"] & P & \text{and} & Y \arrow[ur, bend left, "\lambda_Y"] \arrow[r, "f_Y"] & Q
\end{tikzcd}$$

together with a homotopies $H_X : e_A \circ \lambda_X \simeq f_X$ and $H_Y : e_B \circ \lambda_Y \simeq f_Y$ expressing the homotopy commutativity of the triangles. But there is even more! The homotopies H_X and H_Y must be compatible with each other in the sense described in Section 6.1.

Now we consider going in the other direction. If we are given some of this information, can the remaining data be chosen to fit into a lift and a homotopy?

Theorem 60 *Suppose that in the diagram above, $X \to Y$ is a cofibration and e is a pointwise homotopy equivalence. Then for any choice of*

1. *a lift $\lambda_X : X \to A$ and*
2. *a homotopy $H_X : e_A \circ \lambda_X \simeq f_X$,*

it is possible to find

3. *a lift λ_Y and*
4. *a homotopy $H_Y : e_B \circ \lambda_Y \simeq f_Y$*

so that $\lambda = (\lambda_X, \lambda_Y)$ is a lift of f up to the specific diagram homotopy $(H_X, H_Y) : e \circ \lambda \simeq f$.

The proof is given in Appendix A, but you should feel free to take Theorem 60 for granted. We note, however, that the proof only uses the formal properties of cofibrations, and not any extra topological information. It follows that the proof can be dualized to a statement about fibrations.

PROBLEM 6.19 Show that a map $X \to Y$ is a left nice diagram if and only if it is a cofibration.

Corollary 61 *There is a functorial procedure for producing left nice replacements for single-map diagrams.*

PROBLEM 6.20

(a) Prove Corollary 61.

(b) What is the homotopy colimit of $f : X \to Y$?

6.4.2 Homotopy Pushouts

Here's the key result about prepushout diagrams.

Theorem 62 *Every diagram $C \leftarrow A \to B$ in which both maps are cofibrations is left nice.*

PROBLEM 6.21 Use Theorem 60 to prove Theorem 62.

EXERCISE 6.22 Could a prepushout diagram be left nice without both maps being cofibrations?

Now we can construct left nice replacements. Let $C \leftarrow A \to B$ be a prepushout diagram (which we'll refer to as F). Then we let $\overline{C}$ and $\overline{B}$ be the (reduced, since we are working in the pointed category) mapping cylinders of $A \to C$ and $A \to B$, respectively, and we let $\overline{F}$ denote the prepushout diagram $\overline{C} \leftarrow A \rightarrow \overline{B}$. These diagrams fit into the strictly commutative diagram

$$\begin{tikzcd}
\overline{C} \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\
C & A \arrow[l] \arrow[r] & B
\end{tikzcd}$$

which is a map of diagrams $\overline{F} \rightarrow F$.

EXERCISE 6.23 Verify that this map $\overline{F} \rightarrow F$ is a left nice replacement for F. Also check that the construction is functorial.

Thus we can define $\mathrm{Nice}_L(F) = \overline{F}$; the corresponding homotopy pushout *functor* takes F and returns the categorical pushout of the top row (this is sometimes referred to as the **double mapping cylinder** construction).

PROBLEM 6.24 Determine the homotopy pushouts of the following diagrams.

(a) $* \leftarrow X \rightarrow *$

(b) $* \leftarrow A \rightarrow X$

(c) $X \leftarrow * \rightarrow Y$

The following problem gives a very useful property of prepushout diagrams: a homotopy commutative 'map' of prepushout diagrams can be replaced with an actual map of diagrams that is pointwise homotopy equivalent in $\mathrm{H}\mathcal{T}_*$ to the original diagram.

PROBLEM 6.25 Consider the homotopy commutative diagram

$$\begin{tikzcd}
Z \arrow[d, "f_2"'] & X \arrow[l] \arrow[r] \arrow[d, "f_0"] & Y \arrow[d, "f_1"] \\
C & A \arrow[l] \arrow[r] & B.
\end{tikzcd}$$

This is not a map of diagrams, since it is not commutative.

(a) Assume that $X \rightarrow Y$ and $X \rightarrow Z$ are both cofibrations. Show that there is a diagram map

$$\begin{tikzcd}
Z \arrow[d, "\phi_2"'] & X \arrow[l] \arrow[r] \arrow[d, "f_0"] & Y \arrow[d, "\phi_1"] \\
C & A \arrow[l] \arrow[r] & B
\end{tikzcd}$$

with $\phi_0 \simeq f_0$, $\phi_1 \simeq f_1$ and $\phi_2 \simeq f_2$.

(b) Suppose f_0, f_1 and f_2 are homotopy equivalences. Write F for the top diagram and G for the bottom, and show $\operatorname{hocolim} F \simeq \operatorname{hocolim} G$.

6.4.3 3×3 Diagrams

Next we consider diagrams which have the shape of the following *commutative* diagram

$$\begin{tikzcd}
(\star, \circ) & (\bullet, \circ) \arrow[l] \arrow[r] & (\circ, \circ) \\
(\star, \bullet) \arrow[u] \arrow[d] & (\bullet, \bullet) \arrow[l] \arrow[r] \arrow[u] \arrow[d] & (\circ, \bullet) \arrow[u] \arrow[d] \\
(\star, \star) & (\bullet, \star) \arrow[l] \arrow[r] & (\circ, \star),
\end{tikzcd}$$

which we will refer to as a 3×3 **diagram**.[1]

In our discussion of homotopy pushouts, we started with an especially strong replacement theorem for maps, which allowed us to replace each map separately; and since the replacements agreed with each other at the middle position, we could put them together to replace the whole diagram. We'll pursue a similar strategy for 3×3 diagrams. First, we will find supernice replacements for the individual squares in the diagram, and then we'll piece them together.

So let's concentrate on the upper right hand square

$$\begin{tikzcd}
F(\bullet, \circ) \arrow[r] & F(\circ, \circ) \\
F(\bullet, \bullet) \arrow[u] \arrow[r] & F(\circ, \bullet)
\end{tikzcd} \quad \text{and relabel it} \quad \begin{tikzcd}
B \arrow[r, "i"] & D \\
A \arrow[u, "\beta"] \arrow[r, "\gamma"] & C. \arrow[u, "j"']
\end{tikzcd}$$

We wish to replace this square, in a natural way, with another 'nicer' square. First define $\overline{B} = M_\beta$ and $\overline{C} = M_\gamma$, so that the maps β and γ are replaced with equivalent cofibrations. Now define P to be the pushout of $\overline{C} \leftarrow A \rightarrow \overline{B}$, and let $\phi : P \rightarrow D$ be the unique map. Then, letting $\overline{D} = M_\phi$, we obtain

$$\begin{tikzcd}
\overline{B} \arrow[r] & \overline{D} \\
A \arrow[u] \arrow[r] & \overline{C}, \arrow[u]
\end{tikzcd}$$

which is the diagram we want.

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[1] This shape category is the product of the prepushout diagram with itself.

Exercise 6.26 Check that this diagram is strictly commutative, is pointwise homotopy equivalent to the given square, and that each map in the diagram is a cofibration. Show that the construction is natural, and that the new square comes with a pointwise equivalence to the original square.

If we apply the same procedure to each square in the given diagram, the resulting nice squares have the same maps on the common edges, so that we can put the squares together to form a new diagram $\overline{F}$ and a map $\overline{F} \to F$ which is a pointwise homotopy equivalence.

Theorem 63 *The map $\overline{F} \to F$ is a natural left nice replacement.*

The proof of this theorem is actually fairly straightforward, using what we know about cofibrations.

Problem 6.27

(a) Suppose you are given a lift/homotopy at A; show that it can be extended to lifts/homotopies on $\overline{C} \leftarrow A \to \overline{B}$.

(b) Show that the lift/homotopy on $\overline{C} \leftarrow A \to \overline{B}$ extends to all of $\overline{F}$.

(c) Prove Theorem 63.

Exercise 6.28 Determine the homotopy colimit of the diagram

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
* \arrow[u] \arrow[d] & S^0 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\
* & * \arrow[l] \arrow[r] & *
\end{tikzcd}$$

What happens S^0 is replaced with S^n, or even a general space X?

6.4.4 Telescopes

Recall from Chapter 2 that **telescope diagram** of pointed spaces is a functor $\mathbb{N} \to \mathcal{T}_*$, i.e., a diagram of the form

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n \xrightarrow{f_n} A_{n+1} \xrightarrow{f_{n+1}} \cdots .$$

Theorem 64 *If each map in a telescope diagram is a cofibration, then it is left nice.*

Problem 6.29 Use Theorem 60 to prove Theorem 64.

To construct a left nice replacement of such a diagram, replace the given diagram by a pointwise equivalent one in which all the maps are cofibrations, using the inductive construction

$$\begin{tikzcd}
\cdots \arrow[r] & \overline{A}_{n-1} \arrow[r, "\overline{f}_{n-1}"] \arrow[d, "j_{n-1}"'] & \overline{A}_n \arrow[r, dotted, "\overline{f}_n"] \arrow[d, "j_n"'] & M_{f_n \circ j_n} \arrow[d, dotted] & \\
\cdots \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] & A_{n+1} \arrow[r, "f_{n+1}"] & \cdots .
\end{tikzcd}$$

That is, define $\overline{A}_{n+1} = M_{f_n \circ j_n}$ and use the standard maps.

Problem 6.30 Prove that this construction defines a natural left nice replacement for telescope diagrams.

Exercise 6.31 Formulate and prove the analog of Problem 6.25 for telescopes.

Problem 6.32 Let

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] & A_2 \arrow[r, "f_2"] & \cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] & A_{n+1} \arrow[r, "f_{n+1}"] & \cdots .
\end{tikzcd}$$

be a telescope diagram with homotopy colimit A_∞. If $k \leq l$, write $f_{k,l}$ for the unique map in this diagram from X_k to X_l. Let $r : \mathbb{N} \to \mathbb{N}$ be an increasing function and set up the commutative ladder

$$\begin{tikzcd}
\cdots \arrow[r, "f_{n-1}"] & A_n \arrow[r, "f_n"] \arrow[d, "f_{n,r(n)}"'] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "f_{n+1,r(n+1)}"] & \cdots \\
\cdots \arrow[r, "f_{r(n-1),r(n)}"] & A_{r(n)} \arrow[r, "f_{r(n),r(n+1)}"] & A_{r(n+1)} \arrow[r, "f_{r(n+1),r(n+2)}"] & \cdots .
\end{tikzcd}$$

(a) Show that A_∞ is also a homotopy colimit for the bottom row.

(b) Show that among the induced maps of homotopy colimits is the identity map $A_\infty \to A_\infty$.

(c) Conclude that every induced map is a homotopy equivalence.

(d) Find an example in which the identity map is not the only induced map $A_\infty \to A_\infty$.

6.5 Homotopy Limits

The development of homotopy limits is entirely parallel – and dual – to that of homotopy colimits. Therefore we will briefly summarize the main points of the theory.

6.5.1 Right Nice Diagrams and Homotopy Limits

Instead of pointwise fibrations, we now focus on pointwise *cofibrations*. A map of diagrams $F \to G$ is a **pointwise cofibration** if for each $i \in \mathcal{I}$, the map $F(i) \to G(i)$ is a cofibration in $\mathcal{T}$.

PROBLEM 6.33 Let $\phi : F \to G$ be a map of diagrams. Show that there is a factorization

$$\begin{tikzcd} & \widetilde{G} \arrow[dr, "e"] & \\ F \arrow[ur, "f"] \arrow[rr, "\phi"] & & G \end{tikzcd}$$

of ϕ such that

1. e is a diagram homotopy equivalence
2. f is a pointwise cofibration
3. there is a map $\overline{e} : G \to \widetilde{G}$ such that $\overline{e} \circ e = \mathrm{id}_G$.
4. all this structure is functorial.

HINT Use Theorem 46.

We say that a diagram F is **right nice** if in any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & F \\ B \arrow[ur, dotted, bend right] & \end{tikzcd}$$

in which $A \to B$ is both a pointwise cofibration and a pointwise equivalence, the dotted arrow can be filled in to make the triangle commute.

PROBLEM 6.34 It is equivalent to drop the cofibration condition and ask for the diagram to commute up to diagram homotopy.

Here are some recognition principles for right nice diagrams.

PROBLEM 6.35

(a) State and prove the dual of Theorem 58.

(b) Formulate and prove the dual of Problem 6.11.

We need to know that every diagram can be replaced with a pointwise equivalent nice one, and this is the content of our first main theorem.

Theorem 65 *Let F be a diagram. Then there is a right nice diagram $\overline{F}$ and a pointwise equivalence $F \to \overline{F}$. This can be done functorially.*

Problem 6.36 If $F \to \overline{F}$ and $F \to \widetilde{F}$ are two right nice replacements, then the limits $\lim \overline{F}$ and $\lim \widetilde{F}$ have the same homotopy type.

Now we say that a space X a **homotopy limit** of the diagram F is it is homotopy equivalent to the categorical limit of $\overline{F}$, a right nice replacement for F.

6.5.2 Functoriality of Homotopy Limits

Now we study the effect of a diagram map on the correspoinding homotopy limits.

Problem 6.37 Let $\phi : F \to G$ be a diagram map, and let $F \to \overline{F}$, $G \to \overline{G}$ be right nice approximations.

(a) Show that there is a diagram map $\overline{\phi} : \overline{F} \to \overline{G}$ making the diagram

$$\begin{tikzcd} F \arrow[r, "\phi"] \arrow[d] & G \arrow[d] \\ \overline{F} \arrow[r, "\overline{\phi}"] & \overline{G} \end{tikzcd}$$

commute up to diagram homotopy.

(b) Let X be a homotopy limit for F and let Y be a homotopy limit for G. Show that the map $\phi : F \to G$ induces a map of spaces $X \to Y$.

(c) Show that any two induced maps $i_1 : X_1 \to Y_1$ and $i_2 : X_2 \to Y_2$ are equivalent to one another in the sense that there is a homotopy commutative diagram

$$\begin{tikzcd} X_1 \arrow[r, "i_1"] \arrow[d, "\simeq"'] & Y_1 \arrow[d, "\simeq"] \\ X_2 \arrow[r, "i_2"] & Y_2 \end{tikzcd}$$

(d) Show that any two homotopy limits of F are homotopy equivalent to one another.

In view of Problem 6.37, we can speak of *the* homotopy limit of a diagram; we will sometimes denote it by $\operatorname{holim} F$.

The maps of homotopy limits induced by a diagram map are not unique, but if we agree to use a functorial of nice replacements – which we will denote by $\mathrm{Nice}_R : \mathcal{T}^{\mathcal{I}} \to \mathcal{T}^{\mathcal{I}}$ – then we can define a homotopy limit *functor* by the rule

$$\operatorname{holim} F = \lim \mathrm{Nice}_R(F).$$

Frequently, though, it is more useful to use a right nice replacement that is well suited to the given diagram, or to the problem at hand.

6.5.3 Special Cases: Maps, Pullbacks, 3×3s and Towers

The constructions of right nice replacements of these various diagrams is exactly dual to the constructions given for left nice replacements.

The Key Theorem. The underlying technical theorem, therefore, is the dual of Theorem 60.

Theorem 66 *Suppose that* $X \to Y$ *is a fibration. Then for any* λ_Y *and* H_Y*, it is possible to find* λ_X *and* H_X *so that* (λ_X, λ_Y) *is a extension up to the homotopy* (H_X, H_Y)*.*

Since this theorem is strictly dual to Theorem 60, it could be fair to say that this is not something new that we are taking for granted.

Problem 6.38

(a) Show that a map is right nice if and only if if is a fibration.

(b) Show that the procedure for converting a map to a fibration given in Theorem 49 is actually a natural right nice replacement procedure for single-map diagrams.

Pullbacks. To find nice replacements for prepullback diagrams, we simply replace both maps with fibrations using the procedure we developed in the previous chapter.

Theorem 67 *If* F *is the prepullback diagram* $C \to A \leftarrow B$ *and both maps are fibrations, then* F *is right nice.*

Corollary 68 *There is a functorial construction of right nice replacements for prepullback diagrams.*

Exercise 6.39 Formulate and prove the analog for pullbacks of Problem 6.25.

Exercise 6.40 Study the maps induced by the diagram

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & X \arrow[d, "f"] & * \arrow[l] \arrow[d] \\
* \arrow[r] & Y & *. \arrow[l]
\end{tikzcd}$$

3×3 **Diagrams.** For limits, the relevant diagrams are products of a prepullback diagram with itself. Here's the picture

$$\begin{tikzcd}
(\star, \circ) \arrow[r] \arrow[d] & (\bullet, \circ) \arrow[d] & (\circ, \circ) \arrow[l] \arrow[d] \\
(\star, \bullet) \arrow[r] & (\bullet, \bullet) & (\circ, \bullet) \arrow[l] \\
(\star, \star) \arrow[r] \arrow[u] & (\bullet, \star) \arrow[u] & (\circ, \star), \arrow[l] \arrow[u]
\end{tikzcd}$$

which we will also refer to as a 3×3 **diagram**.[2] In order to find a nice replacement for such a diagram, we dualize the procedure detailed in Section 6.4.3 and obtain a right nice replacement $F \to \overline{F}$.

Proposition 69 *The map of diagrams $\bar{F} \to F$ is a right nice replacement, and it can be done naturally.*

Towers. The dual of a telescope is generally called a tower. It is a diagram with shape

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n \leftarrow n+1 \leftarrow \cdots,$$

which is the category $\mathbb{N}^{\mathrm{op}}$. To construct a nice replacement, simply replace each map with a fibration.

Problem 6.41 Show that this construction does the job.

6.6 Examples of Homotopy Limits and Colimits

In this section we work out some important examples of homotopy limits and colimits.

First we show that a homotopy fibers and cofibers can be thought of as a homotopy limits and colimits, respectively.

Proposition 70 *Let $f : X \to Y$.*

(a) The homotopy pushout of the diagram $* \longleftarrow X \xrightarrow{f} Y$ is cofiber of f.

(b) The homotopy pullback of the diagram $* \longrightarrow Y \xleftarrow{f} X$ is the homotopy fiber of f.

---

[2] This shape category is the product of the prepullback diagram with itself.

Problem 6.42 Prove Proposition 70.

We promised in Section 5.4 to show that no matter how we convert $f : X \to Y$ to a cofibration, the resulting cofiber will be the same. This follows immediately from Proposition 70(b). But even more is true: the cofiber is unchanged if we replace f with another map g which is homotopy equivalent to f.

Problem 6.43 Suppose f and g are homotopy equivalent maps. Show that the cofiber of f is homotopy equivalent to the cofiber of g and the homotopy fiber of f is homotopy equivalent to the homotopy fiber of g.

Problem 6.44 Show that homotopic maps have homotopy equivalent fibers and cofibers.

Here are some other simple, but important, examples.

Problem 6.45

(a) Determine the homotopy pullbacks of the following diagrams

  i. $* \to X \leftarrow *$
  ii. $X \to * \leftarrow Y$.

(b) Determine the homotopy pushouts of the following diagrams.

  i. $* \leftarrow X \to *$
  ii. $X \leftarrow * \to Y$.

Finally, we investigate the homotopy type of an infinite-dimensional CW complex.

Problem 6.46 Let X be a CW complex with skeleta X_n. The skeleta of X form a telescope diagram

$$X_0 \to X_1 \to \cdots \to X_n \to X_{n+1} \to \cdots.$$

Show that X is a homotopy colimit for this diagram.

6.7 Unpointed Homotopy Limits and Colimits

Formally, the definitions of unpointed homotopy colimits and limits are precisely the same as for pointed homotopy colimits and limits. We set them down here and investigate the distinction between unpointed and pointed limits and colimits in some detail.

6.7.1 Unpointed Homotopy Colimits

Given a diagram $F : \mathcal{I} \to \mathcal{T}_\circ$, one finds a left nice replacement $\overline{F} : \mathcal{I} \to \mathcal{T}_\circ$ and we set

$$\operatorname{hocolim}_\circ F = \operatorname{colim} \overline{F}.$$

The difference is in what constitutes a left nice replacement. Now the homotopies in our diagrams are unpointed homotopies, so we need to use unpointed diagram homotopies $F \times I \to G$ instead of the pointed ones $F \rtimes I \to G$. Consequently, when we use the mapping cylinder construction to convert maps to cofibrations, we are forced to use the unreduced mapping cylinder (defined in Section 5.4.

Theorem 71 *Every diagram $F : \mathcal{I} \to \mathcal{T}_\circ$ has a left nice replacement, and hence a homotopy colimit; the homotopy colimit is well defined up to (unpointed) homotopy type.*

We'll take this theorem for granted.

We usually use the notation $\operatorname{hocolim} F$ for the homotopy colimit of a diagram F, whether it is pointed or unpointed. But when there is a possibility of confusion, we will add decorations for clarity, writing $\operatorname{hocolim}_* F$ for the pointed homotopy colimit and $\operatorname{hocolim}_\circ F$ for the unpointed one. Here is a problem that should help you develop an appreciation of the distinction between pointed and unpointed homotopy colimits.

Exercise 6.47 Let $\mathcal{I}$ be any small category, and let $T_\mathcal{I} : \mathcal{I} \to \mathcal{T}$ be the trivial diagram given by $T_\mathcal{I}(i) = *$ for all $i \in \mathcal{I}$.

(a) Show that $T_\mathcal{I}$, considered as a diagram in $\mathcal{T}_*$ is left nice. What is its homotopy colimit?

(b) Give an example of a category $\mathcal{I}$ for which $T_\mathcal{I} : \mathcal{I} \to \mathcal{T}_\circ$ is not left nice. What is the homotopy colimit of $T_\mathcal{I}$?

Hint Use a very small category and discrete spaces.

The unpointed homotopy colimit of $T_\mathcal{I}$ can have interesting topology, and this topology encodes important information about the shape category $\mathcal{I}$. This space is denoted

$$B\mathcal{I} = \operatorname{hocolim}_\mathcal{I} *$$

and called the **classifying space** of $\mathcal{I}$.

6.7.2 Unpointed Homotopy Limits.

The story is very similar for homotopy limits. Given $F : \mathcal{I} \to \mathcal{T}_\circ$, we find a right nice approximation $F \to \overline{F}$, and define $\operatorname{holim} F = \lim \overline{F}$. This is well defined because of the following theorem.

Theorem 72 *Every diagram $F : \mathcal{I} \to \mathcal{T}_\circ$ has a right nice replacement, and hence a homotopy limit; the homotopy colimit is well defined up to (unpointed) homotopy type.*

We'll take this theorem for granted.

Just as for homotopy colimits, the homotopy limit of a trivial diagram is worth studying. Interestingly, the behavior in the unpointed and pointed categories is the same for limits.

Problem 6.48 Show that the constant diagram $T_{\mathcal{I}} : \mathcal{I} \to \mathcal{T}$ is right nice in *both* $\mathcal{T}_\circ$ and $\mathcal{T}_*$. What are the homotopy limits?

Chapter 7

Homotopy Pushout and Pullback Squares

In this chapter, we'll develop some basic tools for manipulating homotopy pushouts and pullbacks. These are basic tricks for manipulating spaces that can be applied in 'any' category where one can do homotopy theory (see Chapter **??** for more detail on this), because they do not involve using target-type constructions as domains, or domain-type constructions as targets. In the next part, we will start to develop theorems to deal with the much more difficult and interesting results that arise when we start to prove things using the special properties enjoyed by our topological categories.

Since homotopy limits and colimits are categorical limits and colimits of related diagrams, many of the properties of limits and colimits remain true for homotopy colimits and limits. The purpose of this chapter is to set down these key properties and derive some important consequences of them.

7.1 Pushouts and Pullbacks of Homotopy Equivalences

Consider the commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

We have seen in Problems 2.20 and 2.26 that, in any category, if the square is a pushout square and $A \to C$ is an isomorphism then $B \to D$ is an isomorphism too; and dually, if the square is a pullback square and $B \to D$ is an isomorphism then so is $A \to C$. Since we are more interested in homotopy equivalences than homeomorphisms, it is natural to inquire whether these statements remain true when the maps are homotopy equivalences rather than isomorphisms.

Exercise 7.1 Give an example of a pushout square where $A \to B$ is a homotopy equivalence but $C \to D$ is not a homotopy equivalence.

Our purpose in this section is to set down, for later reference, a basic and crucial feature of the homotopy theory of topological spaces: the abstact categorical result does hold when isomorphisms are replaced with homotopy equivalences, *provided* the vertical maps are cofibrations (or fibrations).

Theorem 73 *Consider the commutative square*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\ C \arrow[r, "i"] & D. \end{tikzcd}$$

(a) If the square is a pushout square, g is a cofibration, and f is a homotopy equivalence, then i is also a homotopy equivalence.

(b) If the square is a pullback square, h is a fibration, and i is a homotopy equivalence, then f is also a homotopy equivalence.

The proof of Theorem 73 involves the same kinds of ideas as the proof of Theorem 60, and so we give the proof in the same place: Appendix **??**.

7.2 Homotopy Pushout Squares

Analysis of spaces and maps in homotopy theory is often greatly facilitated by the use of the notion of homotopy pushout squares (and, dually, homotopy pullback squares). In this section, we will define homotopy pushout squares and establish some of their most basic properties.

We begin with by investigating how we might compare a given square to a pushout square. If we are given a *strictly commutative* square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

then we may find a left nice approximation $\overline{C} \leftarrow \overline{A} \rightarrow \overline{B}$ for $C \leftarrow A \rightarrow B$. Write $\overline{D}$ for the pushout of $\overline{C} \leftarrow \overline{A} \rightarrow \overline{B}$, so that $\overline{D}$ is the homotopy pushout of $C \leftarrow A \rightarrow B$. Now we can construct the cubical diagram

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr, crossing over] & & \overline{D} \arrow[dd, "\xi"] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] \arrow[from=uu, crossing over] & & D,
\end{tikzcd}$$

showing that there is a natural map $\overline{D} \rightarrow D$. We call maps of this kind *comparison maps* because they allow us to compare the space D with the homotopy colimit of $C \leftarrow A \rightarrow B$.

Exercise 7.2 Explain why the map ξ exists. In what sense is it unique?

Exercise 7.3 Show that for any diagram $F : \mathcal{I} \rightarrow \mathcal{T}$ there is a comparison map $\operatorname{hocolim} F \rightarrow \operatorname{colim} F$, which is unique up to homotopy equivalence of maps.

Problem 7.4 Let $\widetilde{C} \leftarrow \widetilde{A} \rightarrow \widetilde{B}$ be another left nice approximation of the diagram $C \leftarrow A \rightarrow B$, and let $\widetilde{D}$ be its pushout. Show that if $\xi : \overline{D} \rightarrow D$ is a homotopy equivalence, then so is $\widetilde{D} \rightarrow D$.

This is all very nice, but strictly commutative squares are too much to ask for in day-to-day homotopy theory. To define homotopy pushout squares, we need to be able to compare a homotopy commutative square to a pushout square. Suppose, then, that the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

is homotopy commutative. Just as before, we may find a left nice approximation $\overline{C} \leftarrow \overline{A} \rightarrow \overline{A}$ (with pushout $\overline{D}$) for $C \leftarrow A \rightarrow B$ and use it to form the diagram

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& \overline{C} \arrow[rr, crossing over] \arrow[dd] & & \overline{D} \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \text{no map!} \\
& C \arrow[rr] & & D.
\end{tikzcd}$$

This time, however, there is no map $\overline{D} \to D$ that makes the diagram commute, because the bottom square is only commutative up to homotopy: the composites

$$\overline{A} \to \overline{B} \to D \qquad \text{and} \qquad \overline{A} \to \overline{C} \to D$$

are homotopic to each other, but not equal (as far as we know).

We can make some progress, though, if we temporarily assume that the map $\overline{A} \to \overline{B}$ is a cofibration.

Problem 7.5 Show that if $\overline{A} \to \overline{B}$ is a cofibration, then there is a map $\phi : \overline{B} \to D$ so that the diagram

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] \arrow[ddr, bend left, "\phi"] & \\
& \overline{C} \arrow[rr, crossing over] \arrow[dd] & & \overline{D} \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr, dashed] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

commutes up to homotopy, and the solid arrow part of the diagram is strictly commutative. Explain why the map ϕ yields a comparison map $\xi : \overline{D} \to D$ and discuss the uniqueness or nonuniqueness of ξ.

EXERCISE 7.6 Consider the diagram

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[dd, equal] & & CX \arrow[dr] \arrow[dd] \arrow[dddr, bend left=15, "\phi"] & \\
& CX \arrow[rr] \arrow[dd] & & \Sigma X \\
X \arrow[rr] \arrow[dr] & & * \arrow[dr, dashed] & \\
& * \arrow[rr] & & \Sigma X,
\end{tikzcd}$$

and notice that all maps $\phi : CX \to \Sigma X$ are homotopic, since the domain is contractible. Determine the comparison map ξ in the two cases

(a) $\phi = *$, the constant map, and

(b) $\phi : CX \to \Sigma X$ is the map which collapses the bottom of the cone.

Show that if $\Sigma X \not\simeq *$, then these comparison maps are not homotopic, or even homotopy equivalent, to one another.

PROBLEM 7.7 Let $\overline{C} \leftarrow \overline{A} \to \overline{B}$ be your favorite left nice approximation for $C \leftarrow A \to B$, with no assumptions about which maps, if any, are cofibrations. Suppose there is a map $\phi : B \to D$ homotopic to the composition $\overline{B} \to B \to D$ which induces a homotopy equivalence $\xi : \overline{D} \to D$. Let $\widetilde{C} \leftarrow \widetilde{A} \to \widetilde{B}$ be another left nice approximation for $C \leftarrow A \to B$. Show that there is a map $\widetilde{B} \to D$ inducing a homotopy equivalence $\widetilde{\xi} : \widetilde{D} \to D$.

Now we can make our definition.

Definition 74 A homotopy commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

is a **homotopy pushout square** if for any left nice approximation $\overline{C} \leftarrow \overline{A} \to \overline{B}$ of $C \leftarrow A \to B$, there is a map $\phi : \overline{B} \to D$ homotopic to the composite $\overline{B} \to B \to D$ so that the solid arrow part of the homotopy commutative diagram

$$\begin{tikzcd}
\overline{A} \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] \arrow[dddr, bend left, "\phi"] & \\
& \overline{C} \arrow[rr] \arrow[dd] & & \overline{D} \arrow[dd, dotted, "\xi"] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr, dashed] & \\
& C \arrow[rr] & & D
\end{tikzcd}$$

is strictly commutative, and the induced comparison map $\xi : \overline{D} \to D$ is a homotopy equivalence.

This definition has been written to emphasize that the property of being a homotopy pushout square does not depend on making a particular choice of left nice approximation. But, as a practical matter, you should keep in mind that, because of Problem 7.7, it suffices to check that the comparison map is a homotopy equivalence just for your favorite left nice approximation.

Exercise 7.8 Show that if

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a homotopy pushout square, then D is the homotopy pushout of the diagram $C \leftarrow A \to B$.

We end this section with some important examples.

Problem 7.9

(a) Show that the commutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}$$

is a homotopy pushout square.

(b) Prove that $A \to B \to C$ is a cofiber sequence if and only if the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
* \arrow[r] & C
\end{tikzcd}$$

is homotopy pushout square.

Proposition 75 *Consider the homotopy commutative square*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g", "\simeq"'] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

Show that the square is a homotopy pushout square if and only if f is a homotopy equivalence.

PROBLEM 7.10 Prove Proposition 75.

Later, we will produce examples of spaces A such that $A \not\simeq *$ but $\Sigma A \simeq *$. If A is such a space, then the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & * \arrow[d, "f"] \\ * \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout square in which f is a homotopy equivalence, but g is not.

7.3 Recognition and Completion

In order to make use of homotopy pushout squares, it is crucial to be able to recognize them, and it is important to be able to complete any prepushout diagram $C \leftarrow A \rightarrow B$ to a homotopy pushout square. In this section we address both of these needs.

7.3.1 Recognition

We show that a pushout square in which one of the maps is a cofibration is a homotopy pushout square.

Theorem 76 *If in the pushout square*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

the map $A \to C$ is a cofibration, then the square is a homotopy pushout square.

Problem 7.11 Prove Theorem 76 by studying the strictly commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & \overline{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & B \arrow[d] \\
C \arrow[r] & ? \arrow[r] & ??.
\end{tikzcd}$$

Exercise 7.12 Find an example of a homotopy pushout square in which none of the maps is a cofibration. Can you find an example in which the square is also a categorical pushout square?

7.3.2 Completion

Now we show that any set of prepushout data can be completed to a homotopy pushout square.

Theorem 77 *For any prepushout diagram $C \leftarrow A \rightarrow B$ there is a space D and maps $B \rightarrow D$ and $C \rightarrow D$ so that the square*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a homotopy pushout square. This completion is unique up to pointwise equivalence of diagrams in the homotopy category. That is, if $B \rightarrow \widetilde{D}$ and $C \rightarrow \widetilde{D}$ also complete the given diagram to a homotopy pushout square, then there is a homotopy commutative cube

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & B \arrow[dr] \arrow[dd, "\simeq" near end] & \\
& C \arrow[rr] \arrow[dd, "\simeq" near start] & & \widetilde{D} \arrow[dd, "\simeq"] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D.
\end{tikzcd}$$

Problem 7.13

(a) Consider the homotopy commutative cube

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & B \arrow[dr, dashed] \arrow[dd, "\simeq"] & \\
& C \arrow[rr, crossing over] & & D \arrow[dd, "\simeq"] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr, dashed] & \\
& C' \arrow[rr] \arrow[from=uu, crossing over, "\simeq"'] & & D'
\end{tikzcd}$$

whose vertical maps are homotopy equivalences. Suppose the solid arrow part is strictly commutative. Show that if the top face is a homotopy pushout square then the bottom face is also a homotopy pushout square.

(b) Prove Theorem 77.

The result of Problem 7.13(a) will be superseded later by Theorem 84, which is both simpler to state and more powerful.

The completion theorem can be extended to maps of prepushout diagrams.

Problem 7.14 Show that a homotopy commutative diagram

$$\begin{tikzcd}
C \arrow[d] & A \arrow[l] \arrow[r] \arrow[d] & B \arrow[d] \\
Y & X \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

can be completed to a homotopy commutative cube

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd] & & \overline{B} \arrow[dr] \arrow[dd] & \\
& C \arrow[rr, crossing over] & & D \arrow[dd] \\
W \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\
& Y \arrow[rr] \arrow[from=uu, crossing over] & & Z
\end{tikzcd}$$

in which the top and bottom faces are homotopy pushout squares and the map $D \to Z$ is an induced map of homotopy pushouts.

7.4 Homotopy Pullback Squares

In this section we give the basic definitions and theory of homotopy pullback squares. Since this is precisely dual to the discussion of homotopy pushout squares in the previous two sections, this section will have the flavor of a quick summary rather than a detailed study.

We begin by comparing a homotopy commutative square to a pullback diagram. If we are given the homotopy commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D,
\end{tikzcd}$$

then we find a right nice approximation $\overline{C} \to \overline{D} \leftarrow \overline{B}$ with pullback $\overline{A}$ and use it to form the homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dashed] \arrow[dd, dotted, "\zeta"'] \arrow[dddr, bend left=10, "\theta"'] & & B \arrow[dr] \arrow[dd] & \\
& C \arrow[rr] \arrow[dd] & & D \arrow[dd] \\
\overline{A} \arrow[rr] \arrow[dr] & & \overline{B} \arrow[dr] & \\
& \overline{C} \arrow[rr] & & \overline{D}
\end{tikzcd}$$

If there is a map $\theta : A \to \overline{C}$ making the solid arrow part of the diagram strictly commutative, then there is an induced comparison map $\xi : A \to \overline{A}$ from A to the homotopy pullback $\overline{A}$ of $C \to D \leftarrow B$.

Definition 78 The square is a **homotopy pullback square** if for every right nice approximation, a map θ can be found so that the induced comparison map $\zeta : A \to \overline{A}$ is a homotopy equivalence.

As for homotopy pushout squares, we really only need to check this condition for your favorite right nice approximation.

PROBLEM 7.15 Show that if, for your favorite right nice approximation, there exists a θ whose induced comparison map ζ is a homotopy equivalence, then there are such choices for every right nice approximation.

Now we look at some important examples.

Problem 7.16

(a) Show that

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

is a homotopy pullback square.

(b) Show that $F \to E \to B$ is a fibration sequence if and only if

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \\ * \arrow[r] & B \end{tikzcd}$$

is a homotopy pullback square.

Problem 7.17 Consider the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] & B \arrow[d, "f", "\simeq"'] \\ C \arrow[r] & D. \end{tikzcd}$$

(a) Show that the square is a homotopy pullback square if and only if g is a homotopy equivalence.

(b) Find an example of a homotopy pullback square in which g is a homotopy equivalence while f is not.

We finish by addressing the recognition and completion problems.

Problem 7.18 Show that if in the pullback square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & B \arrow[d, "f"] \\ C \arrow[r] & D \end{tikzcd}$$

the map f is a fibration, then the square is a homotopy pullback square.

Problem 7.19 Show that, given prepullback data $C \to D \leftarrow B$, there is a space A and maps $A \to B$, $A \to C$ making the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPB}"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

a homotopy pullback square. Discuss the uniqueness of this square.

7.5 Manipulating Squares

In this section we develop some of the basic formal rules for working with homotopy pushout and pullback squares. We will accumulate a vast collection of applications – with varying levels of importance – of these results in the next section.

7.5.1 Composition of Squares

We begin by adapting Theorems 18 and 19, which concern categorical pushout and pullback squares, to homotopy pushout and pullback squares.

Theorem 79 *Consider the commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "h_1"'] \arrow[dr, phantom, "\textcircled{I}"] & A_2 \arrow[r, "f_2"] \arrow[d, "h_2"'] \arrow[dr, phantom, "\textcircled{II}"] & A_3 \arrow[d, "h_3"] \\
B_1 \arrow[r, "g_1"'] & B_2 \arrow[r, "g_2"'] & B_3,
\end{tikzcd}$$

and denote the outside square by (T).

- *(a) If* (I) *and* (II) *are homotopy pushouts, then* (T) *is also a homotopy pushout.*
- *(b) If* (I) *and* (T) *are homotopy pushouts, then* (II) *is also a homotopy pushout.*
- *(c) If* (I) *and* (II) *are homotopy pullbacks, then* (T) *is also a homotopy pullback.*
- *(d) If* (II) *and* (T) *are homotopy pullbacks, then* (I) *is also a homotopy pullback.*

PROBLEM 7.20 Prove Theorem 79.

7.5.2 3×3 Diagrams

We prove the homotopy theoretic analog of Theorems 20 and 21 and derive some nice consequences. Let us consider the diagram F:

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] \arrow[r] & A_3 \\
B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] \\
C_1 & C_2 \arrow[l] \arrow[r] & C_3
\end{tikzcd}$$

and its left nice approximation $\overline{F}$:

$$\begin{tikzcd}
\overline{A}_1 & \overline{A}_2 \arrow[l] \arrow[r] & \overline{A}_3 \\
\overline{B}_1 \arrow[u] \arrow[d] & \overline{B}_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & \overline{B}_3 \arrow[u] \arrow[d] \\
\overline{C}_1 & \overline{C}_2 \arrow[l] \arrow[r] & \overline{C}_3,
\end{tikzcd}$$

which we construct following the algorithm detailed in Section 6.4.3. Then $Q = \operatorname{colim} \overline{F}$ is the homotopy colimit of the given diagram F.

PROBLEM 7.21

(a) Show that the rows of $\overline{F}$ are left nice replacements for the rows of F. Conclude that the pushouts A, B and C of the rows of $\overline{F}$ are homotopy pushouts of the rows of F.

(b) Show that the induced maps $C \leftarrow \mathcal{A} \rightarrow B$ are cofibrations.

(c) Let $C \leftarrow A \rightarrow B$ be any diagram obtained using maps induced from the given diagram, explain how the diagram gives rise to maps $B \rightarrow Q$ and $C \rightarrow Q$.

(d) Show that with these maps the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\
C \arrow[r] & Q
\end{tikzcd}$$

is a homotopy pushout square.

(e) Verify that all of your arguments are purely formal, and hence dualizable.

You have proved the following theorem.

Theorem 80

(a) *Consider the diagram*

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] \arrow[r] & A_3 \\
B_1 \arrow[u] \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & B_3 \arrow[u] \arrow[d] \\
C_1 & C_2 \arrow[l] \arrow[r] & C_3
\end{tikzcd}$$

whose homotopy colimit is Q. *Taking homotopy pushouts of the rows gives a prepushout diagram* $A \leftarrow B \to C$ *whose maps are induced maps. Taking homotopy pushouts of the columns gives another prepushout diagram* $X \leftarrow Y \to Z$. *Then the squares*

$$\begin{tikzcd}
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & A \arrow[d] \\
C \arrow[r] & Q
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & X \arrow[d] \\
Z \arrow[r] & Q
\end{tikzcd}$$

where all maps are induced from the original diagram, are homotopy pushout squares. In particular, the homotopy pushouts of $A \leftarrow B \to C$ and $X \leftarrow Y \to Z$ *are homotopy equivalent to one another.*

(b) *Consider the diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d] & A_2 \arrow[d] & A_3 \arrow[l] \arrow[d] \\
B_1 \arrow[r] & B_2 & B_3 \arrow[l] \\
C_1 \arrow[r] \arrow[u] & C_2 \arrow[u] & C_3 \arrow[l] \arrow[u]
\end{tikzcd}$$

with homotopy limit Q. *Taking homotopy pullbacks of the rows gives a prepullback diagram* $A \to B \leftarrow C$, *and taking homotopy pullbacks of the columns gives another prepullback diagram* $X \to Y \leftarrow Z$. *Then the squares*

$$\begin{tikzcd}
Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & A \arrow[d] \\
C \arrow[r] & B
\end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}
Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & X \arrow[d] \\
Z \arrow[r] & Y,
\end{tikzcd}$$

where all maps are induced from the original diagram, are homotopy pullback squares. In particular, the homotopy pullbacks of $A \to B \leftarrow C$ *and* $X \to Y \leftarrow Z$ *are homotopy equivalent to one another.*

Here are some nice simple examples.

Problem 7.22 Determine the homotopy colimits of the diagrams

$$\begin{tikzcd}
* & * \arrow[l] \arrow[r] & * \\
* \arrow[u] \arrow[d] & X \arrow[l] \arrow[r] \arrow[u] \arrow[d] & * \arrow[u] \arrow[d] \\
* & * \arrow[l] \arrow[r] & *
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, equal] & X \arrow[r, equal] & X \\
X \arrow[u, equal] \arrow[d, equal] & * \arrow[l] \arrow[r] \arrow[u] \arrow[d] & X \arrow[u, equal] \arrow[d, equal] \\
X \arrow[r, equal] & X \arrow[r, equal] & X.
\end{tikzcd}$$

What is are the duals?

Our final application of Theorem 80 in this section shows that certain combinations of homotopy pushout squares are again homotopy pushout squares, and similarly for homotopy pullbacks.

Lemma 81 *Consider the diagrams*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad \textit{and} \quad
\begin{tikzcd}
W \arrow[r] \arrow[d] & X \arrow[d] \\
Y \arrow[r] & Z.
\end{tikzcd}$$

(a) If they are both homotopy pushout diagrams, then so is

$$\begin{tikzcd}
A \vee W \arrow[r] \arrow[d] & B \vee X \arrow[d] \\
C \vee Y \arrow[r] & D \vee Z.
\end{tikzcd}$$

(b) If they are both homotopy pullback diagrams, then so is

$$\begin{tikzcd}
A \times W \arrow[r] \arrow[d] & B \times X \arrow[d] \\
C \times Y \arrow[r] & D \times Z.
\end{tikzcd}$$

Problem 7.23 Prove Lemma 81.

PROBLEM 7.24

(a) Determine the homotopy pushout of the diagram

$$\begin{tikzcd} B & A \arrow[l, "*"'] \arrow[r, "f"] & C \end{tikzcd}$$

HINT The trivial map $A \to B$ can be viewed as $A \vee * \to * \vee B$.

(b) Determine the homotopy pullback of the diagram $B \xrightarrow{*} A \xleftarrow{f} C$.

(c) What are the fiber and cofiber of a trivial map?

Generalization to Product Diagrams. Let $\mathcal{I}$ and $\mathcal{J}$ be two small categories and choose functorial left nice approximations $\mathrm{Nice}_{\mathcal{I}}$ and $\mathrm{Nice}_{\mathcal{J}}$. Then have corresponding functorial homotopy colimit functors $\mathrm{hocolim}_{\mathcal{I}}$ and $\mathrm{hocolim}_{\mathcal{J}}$. Now let us consider diagrams of the form

$$F : \mathcal{I} \times J \to \mathcal{T}.$$

For each $j \in \mathcal{J}$ we have an $\mathcal{I}$-diagram, and we can form the homotopy colimits to obtain a new $\mathcal{J}$-shaped diagram

$$\mathrm{hocolim}_{\mathcal{I}} F : \mathcal{J} \to \mathcal{T}.$$

Taking homotopy colimit of this diagram gives a space

$$\mathrm{hocolim}_{\mathcal{J}} \mathrm{hocolim}_{\mathcal{I}} F.$$

Theorem 82 *The space* $\mathrm{hocolim}_{\mathcal{J}} \mathrm{hocolim}_{\mathcal{I}} F$ *is the homotopy colimit of* F*. Hence, by symmetry*

$$\mathrm{hocolim}_{\mathcal{I}} \mathrm{hocolim}_{\mathcal{I}} J \simeq \mathrm{hocolim}_{\mathcal{J}} \mathrm{hocolim}_{\mathcal{I}} F.$$

EXERCISE 7.25 Verify that in the special case $\mathcal{I}$ and $\mathcal{J}$ are both the prepushout category $\star \leftarrow \bullet \rightarrow \circ$ this is just Theorem 80.

The proof of Theorem 82 in full gererality depends on certain information about left nice replacements for $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{I} \times \mathcal{J}$ diagrams. Specifically, we need to know that F has a left nice replacement $\overline{F}$ with the property that $\mathrm{hocolim}_{\mathcal{I}} \overline{F}$ is a left nice diagram. Since we do not have that information available, we will take this theorem for granted.

EXERCISE 7.26 Formulate the dual theorem.

7.5.3 Application of Functors

Let's extend Theorem 58 to the homotopy pushout and pullback squares.

PROBLEM 7.27 Let $L, R : \mathcal{T} \to \mathcal{T}$ be an adjoint pair of functors that preserve homotopy equivalences. Suppose further that L preserves cofibrations and R preserves fibrations. Consider the homotopy commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

(a) Show that if $\overline{C} \leftarrow \overline{A} \rightarrow \overline{B}$ is a left nice approximation to $C \leftarrow A \rightarrow B$ with both maps cofibrations, then $L(\overline{C}) \leftarrow L(\overline{A}) \rightarrow L(\overline{B})$ is a left nice approximation to $L(C) \leftarrow L(A) \rightarrow L(B)$.

(b) Show that if the square is a homotopy pushout square, then so is

$$\begin{tikzcd} L(A) \arrow[r] \arrow[d] & L(B) \arrow[d] \\ L(C) \arrow[r] & L(D). \end{tikzcd}$$

(c) State and prove the dual result.

Here are the first important applications.

PROBLEM 7.28 Consider the homotopy commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D. \end{tikzcd}$$

(a) Show that if it is a homotopy pushout square, then so are

$$\begin{tikzcd} A \times X \arrow[r] \arrow[d] & B \times X \arrow[d] \\ C \times X \arrow[r] & D \times X, \end{tikzcd} \quad \begin{tikzcd} A \rtimes X \arrow[r] \arrow[d] & B \rtimes X \arrow[d] \\ C \rtimes X \arrow[r] & D \rtimes X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \wedge X \arrow[r] \arrow[d] & B \wedge X \arrow[d] \\ C \wedge X \arrow[r] & D \wedge X \end{tikzcd}$$

for any space X.

(b) Show that if the original diagram is a homotopy pullback square, then

$$\begin{tikzcd} \mathrm{map}(X, A) \arrow[r] \arrow[d] & \mathrm{map}(X, B) \arrow[d] \\ \mathrm{map}(X, C) \arrow[r] & \mathrm{map}(X, D) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathrm{map}_*(X, A) \arrow[r] \arrow[d] & \mathrm{map}_*(X, A) \arrow[d] \\ \mathrm{map}_*(X, A) \arrow[r] & \mathrm{map}_*(X, A) \end{tikzcd}$$

are also homotopy pullback squares.

7.6 Strong Homotopy Pushout and Pullback Squares

To decide whether the homotopy commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a homotopy pushout square, we need to convert the top map to a cofibration $A \to \overline{B} \to B$ and look at all maps $\overline{B} \to D$ homotopic to the composite $\overline{B} \to B \top D$ and which makes the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & \overline{B} \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

strictly commutative. This yields maps $P \to D$, where P is the categorical pushout of $C \leftarrow A \to \overline{B}$ (and also the homotopy pushout of $C \leftarrow A \to B$). If one of these maps is a homotopy equivalence, then the original square is a homotopy pushout.

This is a very complicated procedure, involving lots of maps to test. Sometimes it is useful to work with a more rigid notion, one where there is only one map to test. The key point is that if the square is strictly commutative, then there is a 'preferred' map $\overline{B} \to D$, namely the actual composition $\overline{B} \to B \to D$. Thus there is a 'preferred' comparison map $P \to D$.

Definition 83 A **strong homotopy pushout square** is a strictly commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in which the natural map $P \to D$ from the homotopy pushout is a homotopy equivalence.

Dually, the sqaure is a **strong homotopy pullback square** if the induced map $A \to Q$ to the homotopy pushout is a homotopy equivalence.

Chapter 8

Many Applications

We use our techniques to derive some crucial results, including the long cofiber and fiber sequences associated to a map $f : X \to Y$. In the next chapter we will derive a vast collection of further consequences.

8.1 Characterization of Homotopy Pushout and Pullback Squares

We show that the property of being a homotopy pushout square is preserved by pointwise equivalences in the homotopy category, and similarly for homotopy pullback squares. This implies a much simpler and more conceptual characterizations of homotopy pushout and pullback squares.

Theorem 84 *Consider the homotopy commutative cube*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "\simeq"'] & & B \arrow[dr] \arrow[dd, "\simeq"'] & \\
& C \arrow[rr] \arrow[dd, "\simeq"'] & & D \arrow[dd, "\simeq"] \\
A' \arrow[rr] \arrow[dr] & & B' \arrow[dr] & \\
& C' \arrow[rr] & & D'
\end{tikzcd}$$

in which the vertical maps are homotopy equivalences. Then

- *(a) the top square is a homotopy pushout square if and only if the bottom square is a homotopy pushout square, and* *(b) the top square is a homotopy pullback square if and only if the bottom is a homotopy pullback square.*

PROBLEM 8.1 Prove Theorem 84.

HINT Flatten the cube into a square planar diagram subdivided into nine smaller squares. Use Theorem 79.

Because of Theorem 84, we can give a new, and conceptually much nicer, characterization of homotopy pushout and pullback squares.

Corollary 85

(a) A homotopy commutative square is a homotopy pushout square if and only if it is pointwise equivalent in the homotopy category to a pushout square in which all four maps are cofibrations.

(b) Dually, a homotopy commutative square is a homotopy pullback square if and only if it is pointwise equivalent in the homotopy category to a pullback square in which all four maps are fibrations.

PROBLEM 8.2 Prove Corollary 85.

8.2 Long Cofiber and Fiber Sequences

In Chapter 6 we showed that a cofiber sequence $A \to B \to C$ gives rise to exact sequences $[A, Y] \leftarrow [B, Y] \leftarrow [C, Y]$. Furthermore, we showed how to put any given map into a cofiber sequence. In view of Problem 7.9, this is done by forming the homotopy pushout diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d, "j"] \\
* \arrow[r] & C.
\end{tikzcd}$$

The key observation is that, since this can be done with *any* map, why not repeat the process with the map j? Here is the diagram that results:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r] \arrow[d, "j"] & * \arrow[d] \\
* \arrow[r] & C \arrow[r, "\partial"] \arrow[d] & A^{(2)} \arrow[r] \arrow[d, "f^{(2)}"] & * \arrow[d] \\
& * \arrow[r] & B^{(2)} \arrow[r, "j^{(2)}"] \arrow[d] & C^{(2)} \arrow[d, "\partial^{(2)}"] \\
& & * \arrow[r] & A^{(3)} \cdots \\
& & & \vdots & \text{etc...}
\end{tikzcd}$$

where the spaces $A^{(2)}, B^{(2)}, C^{(2)}$ and the maps $f^{(2)}$, $g^{(2)}$, etc. are yet to be determined. Notice that the central zigzag of labeled arrows in this diagram is a long cofiber sequence.

Theorem 86 *For any map $f : A \to B$ with cofiber C, there is a long cofiber sequence of the form*

$$A \xrightarrow{f} B \xrightarrow{j} C \xrightarrow{\partial} \Sigma A \xrightarrow{-\Sigma f} \Sigma B \xrightarrow{-\Sigma j} \cdots \longrightarrow \Sigma^n A \xrightarrow{(-1)^n \Sigma^n f} \Sigma^n B \longrightarrow \cdots .$$

PROBLEM 8.3 Using the notation and terminology of the discussion above,

(a) prove that $A^{(2)} \simeq \Sigma A$.

(b) By applying the reasoning of (a) to the map j, show that $B^{(2)} \simeq \Sigma B$.

In Section 6.3 we discussed induced maps between suspensions in some detail. Since the suspension ΣA is the union of two cones, if $f : A \to B$, then extending f to the cones on A and on B yields a map $\phi : \Sigma A \to \Sigma B$. The identification of ϕ depends on which cones we consider to be the top cones and which are the bottom cones. If the map carries the top to the top, then it is Σf; but if it carries the top cone C_+A to the bottom C_-B and also the bottom to the top, then $\phi = -\Sigma f$.

Since our construction involves repeated attachment of cones, we will be unable to identify our maps without deciding on a uniform convention which determines which cone is the top cone. This is the convention we will adopt:

Convention: *The most recently attached cone is the top cone.*

Now we are in position to identify the map $f^{(2)}$.

Problem 8.4

(a) Show that $f^{(2)}$ is induced by the map

$$\begin{tikzcd}
CA \arrow[d] & A \arrow[l] \arrow[r] \arrow[d, "f"] & CB^{(2)} \arrow[d] \\
CC^{(2)} & B \arrow[l] \arrow[r] & CB^{(2)}
\end{tikzcd}$$

of prepushout diagrams. Carefully identify all the maps in this diagram.

(b) Use our convention to identify the map $\Sigma A \to \Sigma B$ induced by this diagram map.

(c) Prove Theorem 86.

Corollary 87 *Let $f : A \to B$ with cofiber C, and let Z be any space. Then there is a long exact sequence*

$$[A, Z] \xleftarrow{f^*} [B, Z] \xleftarrow{j^*} [C, Z] \xleftarrow{\partial^*} [\Sigma A, Z] \longleftarrow \cdots \longleftarrow [\Sigma^n A, Z] \xleftarrow{(-1)^n \Sigma^n f^*} [\Sigma^n B, Z] \leftarrow \cdots .$$

All the terms involving a suspensions are groups, and the maps between them are homomorphisms. The terms involving more than one suspension are abelian groups.

Since this whole discussion just used the formal properties of cofibers and homotopy pushouts, we can dualize the whole thing. So, given a map $f : X \to Y$, we can form the diagram

$$\begin{tikzcd}
\text{...etc} \quad \vdots & & \\
\cdots\ \Omega F \arrow[r] \arrow[d, "-\Omega i"'] & * \arrow[d] & \\
\Omega X \arrow[r, "-\Omega f"] \arrow[d] & \Omega Y \arrow[r] \arrow[d, "\partial"] & * \arrow[d] \\
* \arrow[r] & F \arrow[r, "i"] \arrow[d] & X \arrow[d, "f"] \\
& * \arrow[r] & Y
\end{tikzcd}$$

and thereby derive the dual theorem.

Theorem 88 *For any map f, there is a long fiber sequence of the form*

$$\begin{tikzcd}\cdots \arrow[r] & \Omega^n X \arrow[r, "(-1)^n\Omega^n f"] & \Omega^n Y \arrow[r] & \cdots \arrow[r] & \Omega Y \arrow[r, "\partial"] & F \arrow[r, "i"] & X \arrow[r, "f"] & Y.\end{tikzcd}$$

Corollary 89 *Let $f : X \to Y$ with fiber F, and let A be any space. Then there is a long exact sequence*

$$\begin{tikzcd}\cdots \arrow[r] & {[A,\Omega^n F]} \arrow[r] & {[A,\Omega^n X]} \arrow[r, "(-1)^n\Omega^n f_*"] & {[A,\Omega^n Y]} \arrow[r] & \cdots \arrow[r] & {[A,F]} \arrow[r, "j_*"] & {[A,X]} \arrow[r, "f_*"] & {[A,Y]}.\end{tikzcd}$$

The sets involving at least on loop space are groups, and the maps between them are homomorphisms. The ones involving at least two loops are abelian groups.

As an application, we reprove Proposition **??**.

Problem 8.5 Show that if f is a homotopy equivalence, then its fiber F and cofiber C are both contractible. Are the converses true?

Exercise 8.6 Is our new proof of Proposition **??** actually a new proof? Or do some of the results used in the proof ultimately depend on Proposition **??**, making the new argument circular?

Problem 8.5 suggests we view the cofiber and fiber of f as measures of how far f is from being a homotopy equivalence. If the fiber (or cofiber) is nearly contractible, then we think of f as nearly a homotopy equivalence; but if the fiber (or cofiber) is very far from being contractible, then we think of f as far from being a homotopy equivalence. In Section **??** we will be much more precise about this, and develop concepts that allow us to quantify how 'nearly contractible' a space is.

The construction of these long cofiber and fiber sequences is functorial.

Proposition 90 *A homotopy commutative square*

$$\begin{tikzcd}X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ W \arrow[r, "g"] & Z\end{tikzcd}$$

gives rise to homotopy commutative ladders

$$\begin{tikzcd}X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[r] \arrow[d, "v"'] & C_f \arrow[r] \arrow[d] & \Sigma X \arrow[r, "-\Sigma f"] \arrow[d, "\Sigma u"'] & \Sigma Y \arrow[r] \arrow[d, "\Sigma v"'] & \cdots \\ W \arrow[r, "g"] & Z \arrow[r] & C_g \arrow[r] & \Sigma W \arrow[r, "-\Sigma g"] & \Sigma Z \arrow[r] & \cdots.\end{tikzcd}$$

and, dually,

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega X \arrow[r, "-\Omega f"] \arrow[d, "\Omega u"'] & \Omega Y \arrow[r] \arrow[d, "\Omega v"'] & F_f \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"'] \\
\cdots \arrow[r] & \Omega W \arrow[r, "-\Omega g"] & \Omega Z \arrow[r] & F_g \arrow[r] & W \arrow[r, "g"] & Z.
\end{tikzcd}$$

Problem 8.7

(a) Prove Proposition 90 under the assumption that the square is strictly commutative.

(b) Now suppose that the square is only homotopy commutative. Prove that there is a strictly commutative square

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "u"'] & M_f \arrow[d] \\
Y \arrow[r] & M_g
\end{tikzcd}$$

which is pointwise equivalent, in the homotopy category, to the original square.

(c) Prove Proposition 90.

The naturality of these long sequences can be used to test whether a map is nontrivial.

Problem 8.8 Let $f : X \to Y$ and extend this map to the long cofiber sequence

$$X \to Y \to Z \to \Sigma X \to \Sigma Y \to \Sigma Z \to \cdots.$$

(a) Show that if f is trivial, then there is a map $\Sigma X \to Z$ such that $\Sigma X \to Z \to \Sigma X$ is homotopic to $\mathrm{id}_{\Sigma X}$.

Hint Use the naturality of the long cofiber sequence.

(b) Suppose you have a functor $F : \mathrm{H}\mathcal{T}_* \to \mathrm{AB}\mathcal{G}$ such that $F(Z) = 0$ and $F(\Sigma X) \neq 0$. Show that $f \not\simeq *$.

(c) Formulate and prove the dual results.

Theorem 88 yields a long exact sequence of homotopy groups for a fiber sequence.

Corollary 91 *Let $F \to E \to B$ be a fiber sequence. Then there is a long exact sequence*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \pi_{n-1}(E) \to \cdots.$$

(If $n \geq 1$, then these sets are groups, and if $n \geq 2$, they are abelian.)

Problem 8.9 Prove Corollary 91.

8.3 Commuting (Co)Limits with (Co)Fibers

Our next application concerns the cofibers of the maps in a map of one homotopy pushout square to another.

Theorem 92 *Suppose that in the commutative diagram*

$$\begin{tikzcd}
A_1 \arrow[d] & A_2 \arrow[l] \arrow[r] \arrow[d] & A_3 \arrow[d] & & A \arrow[d] \\
B_1 \arrow[d] & B_2 \arrow[l] \arrow[r] \arrow[d] & B_3 \arrow[d] \arrow[r, "\text{homotopy}", "\text{pushout}"'] & & B \arrow[d] \\
C_1 & C_2 \arrow[l] \arrow[r] & C_3 & & C,
\end{tikzcd}$$

the columns are cofiber sequences. Then the sequence $A \to B \to C$ of homotopy colimits is also a cofiber sequence.

Problem 8.10 Use Theorem 80 to prove Theorem 92.

Problem 8.11 Consider the pushout square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[r, dotted] \arrow[d] & C_f \arrow[d] \\
C \arrow[r, "g"] & D \arrow[r, dotted] & C_g.
\end{tikzcd}$$

Show that the induced map $C_f \to C_g$ is a homotopy equivalence. State and prove the dual statement.

Hint What is the pushout of $C \leftarrow A \to A$?

Generalization to Arbitrary Diagram Shapes. Using Theorem 82, we obtain the following vast generalization of Theorem 92.

Theorem 93 *Let $F \to G \to H$ be a sequence of $\mathcal{I}$-diagrams.*

(a) If the sequence is is pointwise a cofiber sequence, then the induced sequence of homotopy colimits is also a cofiber sequence.

(b) If the sequence is is pointwise a ciber sequence, then the induced sequence of homotopy limits is also a fiber sequence.

Problem 8.12 Prove Theorem 93.

Comparison of Pointed and Unpointed Homotopy Colimits. A diagram of pointed spaces is, after we forget the basepoints, also a diagram of unpointed spaces. We can form the homotopy colimit of the pointed diagram and the unpointed diagram of the unpointed one. How do these two homotopy colimits compare?

In our answer to this question, we have to talk about the same diagram in several different contexts, so we use ostentatiously obvious notation to denote the variants of the diagram in questions. Thus, let $F_* : \mathcal{I} \to \mathcal{T}_*$ be a diagram of pointed spaces; when we forget the basepoints, we obtain the unpointed diagram $F_\circ : \mathcal{I} \to \mathcal{T}_\circ$ of unpointed spaces. The forgetful functor $\mathcal{T}_* \to \mathcal{T}_\circ$ has a left adjoint which attaches a disjoint basepoint; let $F_+ : \mathcal{I} \to \mathcal{T}_*$ be the result of composing this with $F_\circ$.

Now we have two homotopy colmits:

$$\operatorname{hocolim}_\circ F_\circ \qquad \text{and} \qquad \operatorname{hocolim}_* F_*.$$

and we wish to relate them to one another.

The inclusions of the basepoints of the spaces $F(i)$ for $i \in \mathcal{I}$ constitute a natural transformation $T_{\mathcal{I}} \to F_\circ$. Forgetting basepoints and attaching new ones leads to a sequence of diagram maps $(T_{\mathcal{I}})_+ \to F_+ \to F_*$, which, for each i, is a cofiber sequence.

Theorem 94 *There is a natural cofiber sequence*

$$B\mathcal{I} \to \operatorname{hocolim}_\circ F_\circ \to \operatorname{hocolim}_* F_*,$$

for any choice of (cofibrant) basepoint $* \in B\mathcal{I}$.

Corollary 95 *If* $B\mathcal{I} \simeq *$, *then* $\operatorname{hocolim}_\circ F_\circ \simeq \operatorname{hocolim}_* F_*$.

Problem 8.13

(a) Show that if $F : \mathcal{I} \to \mathcal{T}_\circ$, then $\operatorname{hocolim}_* F_+ = (\operatorname{hocolim}_\circ F_\circ)_+$.

(b) Prove Theorem 94 and Corollary 95. Formulate the dual statement.

8.4 Iterated Fibers and Cofibers

We have discussed in Section **??** the idea that the fiber and cofiber of a map can be interpreted as a measure of the deviation of the map from being a homotopy equivalence. In this section we generalize this idea, and produce spaces that measure the deviation of a square from being a homotopy pushout square or a homotopy pullback square.

Consider the homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

Because of Proposition 90 we know that there are induced maps between the cofibers of the rows, as well as between the cofibers of the columns. Thus we obtain the following homotopy commutative 3 × 3 diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & L \arrow[d] \\
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & M \arrow[d, dotted] \\
W \arrow[r] & X \arrow[r, dotted] & \boxed{???}.
\end{tikzcd}$$

How can we fill in the bottom right corner? There are two equally reasonable choices: the cofiber N of $L \to M$ and the cofiber Y of $W \to X$. But which one should we choose?

Exercise 8.14 Show that if the original square is a homotopy pushout square then the cofibers of $L \to M$ and $W \to X$ are both contractible.

Exercise 8.14 suggests that perhaps the cofibers N and Y always have the same homotopy type. That this is generally true is the content of the following theorem, which is sometimes called the Cohen-Moore-Neisendorfer Lemma.

Theorem 96 *Let N be the cofiber of the map $L \to M$ and let Y be the cofiber of the map $W \to X$. Then $N \simeq Y$. Writing Q for this common space, there are induced maps making the square*

$$\begin{tikzcd}
D \arrow[r] \arrow[d] & M \arrow[d] \\
X \arrow[r] & Q
\end{tikzcd}$$

homotopy commutative.

Problem 8.15 Prove Theorem 96. State and prove the dual result.

Hint Use Theorem 80.

These common spaces are known as the **iterated cofiber** and the **iterated fiber**. The idea that the iterated cofiber (or fiber) is a measure of how far the given square is from being a homotopy pushout (or pullback) square is justified by the next problem.

EXERCISE 8.16 Show that if

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

is homotopy commutative, and P is the homotopy pushout of the prepushout part $C \leftarrow A \rightarrow B$, then there are induced maps $P \rightarrow D$. Show that all of the cofibers of these induced maps are homotopy equivalent to one another.

PROBLEM 8.17 This problem elaborates on the previous one, so we'll keep the same notation. Let P denote the homotopy pushout of $C \leftarrow A \rightarrow B$.

(a) Show that the iterated cofiber of the original diagram is homotopy equivalent to the cofiber of any induced map $P \rightarrow D$.

(b) Formulate and prove the dual result.

PROBLEM 8.18 Show that for any composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ there is a fiber sequence

$$F_f \rightarrow F_{g\circ f} \rightarrow F_g.$$

PROBLEM 8.19 If $f : X \rightarrow Y$, the **graph** of f is the map $\mathrm{Graph}(f) : X \rightarrow X \times Y$ given by $x \mapsto (x, f(x))$. Determine the homotopy fiber of $\mathrm{Graph}(f)$.[1]

8.5 Mayer-Vietoris Sequences

If X is constructed as the homotopy pushout of $C \leftarrow A \rightarrow B$, we can view X as the union of B and C, where B and C intersect along A. The following result is the topological underpinning of the Mayer-Vietoris sequences, which are algebraic tools for understanding the space X in terms of its pieces A, B and C.

PROBLEM 8.20 Let

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

---

[1] This problem came up in conversation with Ran Levi, who attributed it to Fred Cohen.

be a homotopy pushout square. Show that there is a cofiber sequence

$$B \vee C \xrightarrow{(j,g)} D \xrightarrow{\partial} \Sigma A.$$

This cofiber sequence is known as the **Mayer-Vietoris cofiber sequence**. Formulate and prove the dual.

EXERCISE 8.21 Extend the Mayer-Vietoris sequence to the right to get the long sequence

$$B \vee C \xrightarrow{(j,g)} D \xrightarrow{\partial} \Sigma A \xrightarrow{\theta} \Sigma B \vee \Sigma C \longrightarrow \cdots.$$

identify the map θ in terms of the maps f, g, i and j from the original square.

It may be a bit frustrating that we cannot start our Mayer-Vietoris sequence one step to the left. In fact we can do that if the space in the upper left corner is a suspension.

PROBLEM 8.22 Show that the following are equivalent:

1. There is a cofiber sequence $\Sigma X \xrightarrow{\alpha-\beta} A \vee B \xrightarrow{(i,j)} Y$.
2. There is a homotopy pushout square

$$\begin{tikzcd}
\Sigma X \arrow[r, "\alpha"] \arrow[d, "\beta"] & A \arrow[d, "i"] \\
B \arrow[r, "j"] & Y.
\end{tikzcd}$$

3. State and prove the dual statement.

HINT Find maps (not homotopy classes!) $CX \to A$ and $CX \to B$ that give a map of prepushout squares

$$\begin{tikzcd}
CX \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & CX \arrow[d] \\
B & * \arrow[l] \arrow[r] & A
\end{tikzcd}$$

which induces $\alpha - \beta$.

8.6 H Spaces and coH Spaces

There are many spaces which have part of the structure needed to make them group objects, but not all of it. These spaces are called *H spaces*; and the dual – spaces that are not quite cogroup objects – are called *coH spaces.*

8.6.1 H-Spaces

The most basic idea is that of an **H-sapce**. An H-space is a space $X \in \mathcal{T}_*$ which has a map $\mu : X \times X \to X$, called its **multiplication**, that makes the diagram

$$\begin{tikzcd}
X \vee X \arrow[d] \arrow[dr, bend left, "\nabla"] & \\
X \times X \arrow[r, "\mu"] & X
\end{tikzcd}$$

commute up to homotopy. An H-space X is called **associative** if the square

$$\begin{tikzcd}
X \times X \times X \arrow[r, "\mathrm{id}_X \times \mu"] \arrow[d, "\mu \times \mathrm{id}_X"'] & X \times X \arrow[d, "\mu"] \\
X \times X \arrow[r, "\mu"] & X
\end{tikzcd}$$

commutes up to homotopy. It is called a **commutative** H-space if the diagram

$$\begin{tikzcd}
X \times X \arrow[rr, "T"] \arrow[dr, "\mu"'] & & X \times X \arrow[dl, "\mu"] \\
& X &
\end{tikzcd}$$

commutes up to homotopy (where T is the twist map). If $f : X \to Y$ where X and Y are H-spaces, then we say that f is an **H-map** (or a homomorphism) if the square

$$\begin{tikzcd}
X \times X \arrow[r, "f \times f"] \arrow[d, "\mu"'] & Y \times Y \arrow[d, "\mu"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes up to homotopy.

PROBLEM 8.23 Show that if X is an H-space, then $[A, X]$ has a multiplication that is natural as a functor of A. Show that if $f : X \to Y$ is an H-map, then the natural transformation $f_* : [\,?\,, X] \to [\,?\,, Y]$ respects multiplication.

PROBLEM 8.24 Show that if X is a retract of Y (up to homotopy) and Y is an H-space, then X is also an H-space.

EXERCISE 8.25 Suppose that X is a homotopy retract of Y.

(a) What can you say about X if Y is commutative? or if Y is associative?

(b) Is there necessarily an H-map from X to Y or from Y to X?

PROBLEM 8.26 Show that X is an H-space if and only if in any diagram of the form

$$\begin{tikzcd} A \vee B \arrow[r, "{(f,g)}"] \arrow[d] & X \\ A \times B \arrow[ur, dotted] & \end{tikzcd}$$

the dotted arrow can be filled in to make the diagram commute up to homotopy. Can it be filled in to commute on the nose?

8.6.2 Co-H-Spaces

Now we turn to the dual notion. A **co-H-space** is a space X with a map $\phi : X \to X \vee X$ making the diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[dr, bend right, "\Delta"'] & X \vee X \arrow[d] \\ & X \times X \end{tikzcd}$$

commute up to homotopy.

EXERCISE 8.27 Formulate definitions for commutative and associative co-H-spaces.

PROBLEM 8.28 Show that if X is a co-H-space, then $[X, ?\,]$ takes its values in sets with unital multiplications.

PROBLEM 8.29 Formulate and prove the dual of Problem 8.26.

PROBLEM 8.30 Show that a retract of a co-H-space is also a co-H-space.

We know that a suspension is cogrouplike, so it is automatically a co-H-space. Our next problem explores the extent to which a co-H-space must be like a suspension.

We need another definition. We say a subset $U \subseteq X$ is a **categorical subset** if the inclusion map $i : U \hookrightarrow X$ is homotopic to a constant map. A **categorical cover** of X is a cover

$$X = U_0 \cup U_1 \cup \cdots \cup U_n$$

in which each set $U_k \subseteq K$ is a closed categorical subset X. The least n for which X has such a cover is called the **Lusternik-Schnirelmann category** of X (usually, simply the **category** of X); it is denoted $\mathrm{cat}(X)$.

EXERCISE 8.31 Show that $\mathrm{cat}(X) = 0$ if and only if $X \simeq *$ and that for any space X, $\mathrm{cat}(\Sigma X) \leq 1$.

PROBLEM 8.32

(a) Suppose X is a co-H-space with comultiplication $\phi : X \to X \vee X$. Let $A = \phi^{-1}(X \times *)$ and $B = \phi^{-1}(* \times X)$. Show that A and B are categorical, and conclude that $\mathrm{cat}(X) \leq 1$.

(b) Show that if $\mathrm{cat}(X) \leq 1$ then X is a co-H-space.

(c) Suppose the inclusions $A \cap B \hookrightarrow A$ and $A \cap B \hookrightarrow B$ are cofibrations. Show that if $\mathrm{cat}(X) \leq 1$ then X is a retract of a suspension.

HINT Use Problem 2.28 and the diagram

$$\begin{tikzcd}
B \arrow[d] & A \cap B \arrow[l] \arrow[r] \arrow[d, equal] & A \arrow[d] \\
CB & A \cap B \arrow[l] \arrow[r] & CA.
\end{tikzcd}$$

It is worth stating the conclusion of your work as a proposition.

Proposition 97 *A normal space X is a co-H-space if and only if X is a retract of a suspension.*

PROBLEM 8.33 Show that if X and Y are coH spaces, then $X \wedge Y$ is a cocommutative coH space.

PROBLEM 8.34 If A is a co-H-space and X is an H-space, then the set $[A, X]$ inherits two multiplications: one from A and one from X. Show that they coincide.

PROBLEM 8.35 Let X be an H-space with multiplication $\mu : X \times X \to X$. The **shear map** $s : X \times X \to X \times X$ is the map $s = (\mathrm{pr}_1, \mu)$. Using the canonical identification $\pi_*(X \times X) \cong \pi_*(X) \times \pi_*(X)$,

(a) identify the induced map μ_*

(b) identify the induced map s_*.

8.7 An Introduction to Moore Spaces

In this section, we'll determine some mapping sets $[M, S]$. This is moderately interesting in its own right, but it also shows how knowledge of the homotopy groups of spheres $\pi_n(S^m)$ can be useful in solving other problems, and also it shows some important algebraic constructions inserting themselves into our nice clean topological world.

8.7.1 Definition of Moore Spaces

To begin, let's consider the group $\pi_n(S^n) = [S^n, S^n]$. This group has two elements that we can name: $*$ and id_{S^n}. Since we have not yet shown that $S^n \not\simeq *$, it is conceivable that these are actually two different names for the same element. But this is not the case, and the group $\pi_n(S^n)$ is given by the following theorem (notice that we are not using any notation to distinguish between a map and its homotopy class – this is the usual custom in homotopy theory).

Theorem 98 *$\pi_n(S^n) \cong \mathbb{Z}$.*

This is a crucial computation in homotopy theory, and we do not yet have the tools needed to prove it. It will appear later – with a proof – as Theorem **??**; for the purposes of this section, though, you should take it for granted.

Knowing the group $\pi_n(S^n)$ up to isomorphism is nice, but it is much more useful to know what it is generated by.

PROBLEM 8.36 Let $g : S^n \to S^n$ be a map whose homotopy class generates $\pi_n(S^n)$. Consider the map $g_* : \pi_n(S^n) \to \pi_n(S^n)$ induced by g.

(a) Show that $g_*(\mathrm{id}_{S^n}) = g$, and conclude that g_* is an isomorphism.

(b) Show that $g_*(g) = \pm g$.

(c) Show that $g = \pm\mathrm{id}_{S^n}$.

You have shown that if $\pi_n(S^n) \cong \mathbb{Z}$, then $\pi_n(S^n)$ is generated by id_{S^n}. Because of Theorem 98, this implies that any map $f : S^n \to S^n$ is homotopic to $a \cdot \mathrm{id}_{S^n}$ for a unique $a \in \mathbb{Z}$; this number a is called the **degree** of f, and denoted $\deg(f)$. We will refer to this map simply as a. Thus $2 : S^n \to S^n$ denotes the unique homotopy class with degree 2, a homotopy class that is represented by $2 \cdot \mathrm{id}_{S^n} = \mathrm{id}_{S^n} + \mathrm{id}_{S^n}$.

PROBLEM 8.37 Let $a \in \mathbb{N}$.

(a) Write down the composition of maps which defines

$$a \cdot \mathrm{id}_{S^n} = \overbrace{\mathrm{id}_{S^n} + \mathrm{id}_{S^n} + \cdots + \mathrm{id}_{S^n}}^{a \text{ terms}} \in [S^n, S^n].$$

(b) Write down the composition of maps which defines $a \cdot f \in \pi_n(X)$.

(c) Show that $a^* : [S^n, X] \to [S^n, X]$ is given by $a^*(f) = a \cdot f$.

(d) Show that $\Sigma(a \cdot \mathrm{id}_{S^n}) = a \cdot \mathrm{id}_{S^{n+1}}$.

Observe that when $n \geq 2$, $\pi_n(X)$ is an abelian group, so α_* is a homomorphism; but when $n = 1$ all bets are off.

Exercise 8.38 Let $\alpha \in \pi_n(X)$, with $n \geq 2$.

(a) Suppose α is a homotopy equivalence. What can you say about $\deg(\alpha)$?

(b) Suppose α and β are homotopy equivalent maps. How are $\deg(\alpha)$ and $\deg(\beta)$ related?

Using the concept of degree, we can define new collection of spaces. The **Moore space** for the group $\mathbb{Z}/a$ in dimension n is the homotopy pushout in the square

$$\begin{tikzcd}
S^n \arrow[r, "\alpha"] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & S^n \arrow[d] \\
* \arrow[r] & M(\mathbb{Z}/a, n),
\end{tikzcd}$$

where α has degree a.

Problem 8.39 Let $n \geq 2$, and let $a \in \mathbb{N}$.

(a) Find a space N such that $M(\mathbb{Z}/a, n) \simeq \Sigma N$. For which values of n is $M(\mathbb{Z}/a, n)$ a *double* suspension?

(b) Conclude that the functor $[M(\mathbb{Z}/a, n), ?\,]$ takes its values in the category $\mathcal{G}$; for which n does it land in $\text{AB}\,\mathcal{G}$?

8.7.2 Maps Out of Moore Spaces

Now we'll describe the group $[M(\mathbb{Z}/a, n), X]$ in terms of the homotopy groups $\pi_*(X)$ and some some concepts from homological algebra. If G is an abelian group, then the **tensor product** of G with $\mathbb{Z}/a$ is the group

$$G \otimes \mathbb{Z}/a = G/a \cdot G.$$

The **torsion** of G with $\mathbb{Z}/a$ is the group

$$\text{Tor}(G, \mathbb{Z}/a) = \{g \in G \,|\, a \cdot g = 0\}.$$

Tensor and Tor are defined for any pair of abelian groups, or even more generally for any two modules M and N over the same ring R. These functors which play a major role in homological algebra, and one of the points of these problems is that when you study homotopy theory, you are forced to study homological algebra, too.

Exercise 8.40 Suppose $A \to B \to C \to D$ is an exact sequence, and write

$$Q = \text{Im}(B \to C) = \ker(C \to D).$$

Show that the sequence can be expanded to

$$\begin{tikzcd}
A \arrow[r] & B \arrow[rr] \arrow[dr] & & C \arrow[r] & D \\
& & Q \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0, &
\end{tikzcd}$$

where the sequences $A \to B \to Q \to 0$ and $0 \to Q \to C \to D$ are exact.

PROBLEM 8.41 Let $a, n \in \mathbb{N}$.

(a) Show that there is an exact sequence

$$\pi_{n+1}(X) \xrightarrow{a^*} \pi_{n+1}(X) \longrightarrow [M_n(\mathbb{Z}/a), X] \longrightarrow \pi_n(X) \xrightarrow{a^*} \pi_n(X).$$

(b) Show that there is an exact sequence

$$0 \longrightarrow \pi_{n+1}(X) \otimes \mathbb{Z}/a \longrightarrow [M(\mathbb{Z}/a, n), X] \longrightarrow \mathrm{Tor}(\pi_n(X), \mathbb{Z}/a) \longrightarrow 0.$$

(c) If $n > 7$, then the first few homotopy groups of S^n are given by

k	0	1	2	3	4	5	6
$\pi_{n+k}(S^n)$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$

Use these values to determine the groups $[M_{n+k}(\mathbb{Z}/p), S^n]$ for all prime numbers p and all $k \leq 5$.

(d) Is it possible that $M(\mathbb{Z}/a, n)$ is contractible?

8.8 A Diagram in h$\mathcal{T}_*$ With No Pushout

Where does this section belong?

We will show that there is a diagram in h$\mathcal{T}_*$ with no pushout, using the following *as yet unproved* information about the homotopy groups of spheres:

1. for $n \geq 1$, $\pi_n(S^n) = \mathbb{Z} \cdot [\mathrm{id}]$
2. for all $k < n$, $\pi_k(S^n) = 0$.

This will be proved later, after we establish the topological inputs to homotopy theory.

PROBLEM 8.42 A prepushout diagram $C \leftarrow A \to B$ in h$\mathcal{T}_*$ is the image of many prepushout diagrams in $C \leftarrow A \to B$ in $\mathcal{T}_*$. Show that all of these preimage diagrams have homotopy equivalent homotopy pushouts.

Because of Problem 8.42, we can meaningfully talk about the homotopy pushout of a prepushout diagram in $\mathrm{H}\mathcal{T}_*$.

PROBLEM 8.43 Let $C \leftarrow A \rightarrow B$ be a prepushout diagram in $\mathrm{H}\mathcal{T}_*$, that has a pushout P.

(a) Show that for any choice of homotopy pushout D there is a uniquely defined comparison map $i : P \to D$ in $\mathrm{H}\mathcal{T}_*$.

(b) Show that i has a left homotopy inverse $r : D \to P$ so that P is a retract of D in $\mathrm{H}\mathcal{T}_*$.

Now we use our (very limited) knowledge of homotopy groups to get some control over retracts of spheres.

PROBLEM 8.44 Suppose P is a homotopy retract of S^n (for $n = 1, 2, 3$). Show that either $P \simeq *$ or $P \simeq S^n$.

HINT Let $r : S^n \to P$ and $j : P \to S^n$ such that $r \circ j \simeq \mathrm{id}_P$. Show that $j \circ r \simeq \mathrm{id}_{S^n}$.

Recall from Section **??** that the Moore space $M(\mathbb{Z}/a, n)$ is the cofiber of the degree a map $\mathbf{a} : S^n \to S^n$. It sits in a long cofiber sequence

$$\begin{tikzcd} S^n \arrow[r, "\mathbf{a}"] & S^n \arrow[r, "j"] & M(\mathbb{Z}/a, n) \arrow[r, "\partial"] & S^{n+1} \arrow[r, "\mathrm{a}"] & S^{n+1} \arrow[r] & \cdots . \end{tikzcd}$$

PROBLEM 8.45 Let $a \in \mathbb{N}$ with $a \geq 2$.

(a) Show $M(\mathbb{Z}/a, 2) \not\simeq *$.

(b) Show that the induced map $\partial^* : [S^3, S^3] \to [M(\mathbb{Z}/a, 2), S^3]$ is not injective.

(c) Show that the prepushout diagram $\begin{tikzcd} * & S^2 \arrow[l] \arrow[r, "j"] & M(\mathbb{Z}/a, 2) \end{tikzcd}$ has no pushout in $\mathrm{H}\mathcal{T}_*$.

8.9 Homotopy Type of Joins and Products

In this section we introduce a construction called the **join** of two spaces. Joins are defined by as a homotopy pushout, but surprisingly they appear quite frequently in the study of homotopy fibers of a many important maps. We determine the homotopy type of the join in terms of smash products and suspensions. Then we show how the join leads to a splitting of $\Sigma(X \times Y)$ as a wedge sum of other spaces. Then we turn to the study of the homotopy type of a product of two mapping cones, which leads us to the Whitehead product.

The Join of Two Spaces. The join of X and Y is defined up to homotopy type as the homotopy pushout in the square

$$\begin{tikzcd} X \times Y \arrow[r, "\mathrm{pr}_X"] \arrow[d, "\mathrm{pr}_Y"'] \arrow[dr, phantom, "\text{HPO}"] & X \arrow[d] \\ Y \arrow[r] & X{*}Y. \end{tikzcd}$$

We define it – up to homeomorphism – as the *standard* homotopy pushout of the diagram $Y \leftarrow X \times Y \rightarrow X$.

EXERCISE 8.46 Show that $X * Y$ is homeomorphic to the quotient of $X \times I \times Y$ by the equivalence relation given by

$$(x, 0, y) \sim (x, 0, y') \qquad\qquad (x, 1, y) \sim (x', 1, y)$$

for any $x, x' \in X$ and $y, y' \in Y$.

PROBLEM 8.47

(a) Show that the square

$$\begin{tikzcd} X \vee Y \arrow[r, "(\mathrm{id}_X{,}*)"] \arrow[d, "(*{,}\mathrm{id}_Y)"'] & X \arrow[d] \\ Y \arrow[r] & * \end{tikzcd}$$

is a homotopy pushout square. Is it a categorical pushout square?

(b) Show that $X * Y \simeq \Sigma(X \wedge Y)$.

EXERCISE 8.48 Are the spaces $X * Y$ and $\Sigma(X \wedge Y)$ homeomorphic?

Splittings of Products. We turn now to a surprising and extremely useful decomposition of the suspension of a product as a wedge of spaces.

PROBLEM 8.49

(a) Show that the vertical map $X \to X * Y$ (in the homotopy pushout square that defines the join) is the trivial map $*$.

(b) Use a Mayer-Vietoris sequence to express the space $\Sigma(X \times Y)$ as a wedge of other spaces.

(c) Identify the maps

$$\Sigma(\mathrm{in}_X) : \Sigma(X) \hookrightarrow \Sigma(X \times Y) \qquad \text{and} \qquad \Sigma(\mathrm{pr}_X) : \Sigma(X \times Y) \to \Sigma X$$

in terms of your answer to part (b)

(d) Determine the cofiber of the projection map $\mathrm{pr}_Y : X \times Y \to Y$.

The formula you derived in Problem 8.49(b) shows that products **split** as a wedge after one suspension. Some spaces require more than one suspension to split, and still others never split.

Problem 8.50 Using Problem 8.49(b), write down a splitting of the n-fold product $\Sigma(X_1 \times X_2 \times \cdots \times X_n)$.

It is easy enough to dualize this. The smash product is the cofiber of $X \vee Y \to X \times Y$, so the dual is the fiber of $X \vee Y \to X \times Y$.

One of the first notations for the smash product was the musical 'sharp' symbol: $X \sharp Y$. Authors who used this notation were naturally led to use the 'flat' symbol for the dual operation, and so you will sometimes see the homotopy fiber of $X \vee Y \to X \times Y$ denoted $X \flat Y$ (and referred to as the 'flat product'). Both of these notations have fallen out of use. Presently, you will determine the homotopy type of this fiber in terms of the basic constructions we already know, so we won't need any permanent notation for this space. I think that 'flat' notation seems to indicate an asymmetrical dependence on X and Y, so I will use the temporary, but symmetrical, notation $X \diamond Y$ for this homotopy fiber.

Problem 8.51

(a) Determine the homotopy type of $X \diamond Y$.

(b) Determine the fiber of $\mathrm{in}_1 : X \to X \vee Y$.

Products of Mapping Cones. From our discussion of the cellular structure of a product of CW complexes in Chapter **??**, we know that $S^n \times S^m = (S^n \vee S^m) \cup_w D^{n+m}$ for some map $w : S^{n+m-1} \to S^n \vee S^m$. In this section, we derive a far reaching generalization of this decomposition, one that sheds considerable light on the structure of products of spaces.

Given two maps, $f : A \to X$ and let $g : B \to Y$, we may form the mapping cones $C_f = X \cup_f CA$ and $C_g = Y \cup_g CB$. Inside the product $C_f \times C_g$ we have a subspace $T(f,g)$ defined by the pushout square

$$\begin{tikzcd}
X \times Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & X \times C_g \arrow[d] \\
C_f \times Y \arrow[r] & T(f,g).
\end{tikzcd}$$

We will determine precisely how $C_f \times C_g$ is constructed from $T(f,g)$.

Exercise 8.52 Let $f : A \to X$ and let $g : B \to Y$.

(a) Show that the inclusion $T(f,g) \hookrightarrow C_f \times C_g$ is a cofibration. What is its cofiber?

(b) Show that $T(\mathrm{id}_A, \mathrm{id}_B) = A * B$.

(c) Let $f : A \to *$ and $g : B \to *$; then show $T(f,g) = \Sigma A \vee \Sigma B$.

Here is our main result.

Proposition 99 *Let $f : A \to X$ and let $g : B \to Y$. Then there is a cofiber sequence*

$$A * B \to T(f,g) \to C_f \times C_g$$

which is functorial in both f and g.

EXERCISE 8.53 Write out explicitly what it means for the sequence to be functorial in f and g. What categories are involved? What functors?

PROBLEM 8.54 Consider the diagram

$$\begin{tikzcd}
A \times Y \arrow[rr] \arrow[dr] \arrow[dd] & & X \times Y \arrow[dr] \arrow[dd] & \\
& A \times C_g \arrow[rr] \arrow[dd] & & X \times C_g \arrow[dd] \\
CA \times Y \arrow[rr] \arrow[dr] & & C_f \times Y \arrow[dr] & \\
& T(\mathrm{id}_A, f) \arrow[rr] \arrow[d] & & T(f,g) \arrow[d] \\
& CA \times C_g \arrow[rr] & & C_f \times C_g.
\end{tikzcd}$$

(a) Show that the lower square is a pushout square.

(b) Show that there is a pushout square

$$\begin{tikzcd}
A * B \arrow[r, "w"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & T(f,g) \arrow[d] \\
CA \times CB \arrow[r] & C_f \times C_g.
\end{tikzcd}$$

(c) Prove Proposition 99.

EXERCISE 8.55 Show that there is a commutative square

$$\begin{tikzcd}
A * B \arrow[r, equal] \arrow[d] & A * B \arrow[d] \\
C(A * B) \arrow[r, "\cong"] & CA \times CB,
\end{tikzcd}$$

and conclude that $C_f \times C_g \cong C_w$, where $w : A * B \to T(f,g)$ is the map from Problem 8.54(b).

PROBLEM 8.56 Let $f : A \to X$ and let $g : B \to Y$.

(a) Determine the cofiber of the inclusion $X \times Y \hookrightarrow T(f,g)$.

(b) Let Q be the cofiber of $X \times Y \hookrightarrow C_f \times C_g$. Show that there is a cofiber sequence

$$A * B \longrightarrow (\Sigma A \rtimes Y) \vee (X \ltimes \Sigma B) \longrightarrow Q.$$

(c) Let R be the cofiber of $X \wedge Y \hookrightarrow C_f \wedge C_g$, and show that there is a cofiber sequence

$$A * B \longrightarrow (\Sigma A \wedge Y) \vee (X \wedge \Sigma B) \longrightarrow R.$$

Whitehead Products. Our decomposition is particularly nice when $f : A \to *$ and $g : B \to *$, for then it asserts the existence of a cofiber sequence

$$A * B \xrightarrow{w} \Sigma A \vee \Sigma B \longrightarrow \Sigma A \times \Sigma B,$$

and we understand all the spaces in the sequence. The map $w : A * B \to \Sigma A \vee \Sigma B$ in this sequence is an example of a (generalized) **Whitehead product**. More generally, if $\alpha : \Sigma A \to X$ and $\beta : \Sigma B \to X$, then the generalized Whitehead product of α and β is the map $[\alpha, \beta]$ defined by the composition

$$\begin{tikzcd} A * B \arrow[rr, "{[\alpha,\beta]}"] \arrow[dr, "w"'] & & X \\ & \Sigma A \vee \Sigma B. \arrow[ur, "{(\alpha,\beta)}"'] & \end{tikzcd}$$

Thus $w = [\mathrm{id}_A, \mathrm{id}_B]$.

PROBLEM 8.57 Let $\alpha \in [\Sigma A, X]$ and $\beta \in [\Sigma B, X]$.

(a) Show that the Whitehead product $[\alpha, \beta]$ is well defined up to homotopy.

(b) Show that if X is an H space, then $[\alpha, \beta] = *$.

Whitehead products were first defined for homotopy classes $\alpha, \beta \in \pi_*(X)$. Let's look at that case in more detail.

PROBLEM 8.58

(a) Suppose $A = S^n$ and $B = S^m$. What homotopy set does $[f, g]$ belong to?

HINT What is $A * B$?

(b) Show that in the case $A = B = S^0$, $[\alpha, \beta]$ is the commutator $\alpha\beta\alpha^{-1}\beta^{-1}$.

HINT View $S^1 \times S^1$ as a square with parallel sides identified.

prove the graded jacobi identity?

8.10 Cone Decompositions and Lusternik-Schnirelmann Category

Introduction.... Needs Revision!

Cone Decompositions. A map $f : X \to Y$ is called a **principal cofibration** if there is a space A and a map $A \to X$ such that the sequence $A \to X \to Y$ is a cofiber sequence. Because of the coaction of ΣA on Y, These maps are particularly easy to work with on the domain side.

Exercise 8.59 Show that the suspension of any map is a principal cofibration.

If f is not a principal cofibration, then we may try to understand it by expressing it – up to homotopy equivalence – as a composition of a finite number of principal cofibrations. More precisely, we may try to find a homotopy commutative diagram of the form

$$\begin{tikzcd}
A_0 \arrow[d] & A_1 \arrow[d] & & A_{n-1} \arrow[d] & \\
X_0 \arrow[r] \arrow[d, "\simeq"'] & X_1 \arrow[r] & \cdots \arrow[r] & X_{n-1} \arrow[r] & X_n \arrow[d, "\simeq"] \\
X \arrow[rrrr, "f"] & & & & Y
\end{tikzcd}$$

in which each sequence $A_k \to X_k \to X_{k+1}$ is a cofiber sequence. Such a diagram is called a **cone decomposition** of f with **length** n. We consider $* \to X$ to be a cone decomposition of id_X with length 0.

It can be useful to think of a cone decomposition of f as a recipe for building Y from X using the basic pieces $A_0, A_1, \ldots, A_{n-1}$. Sometimes it happens that we want to exert some control over the pieces involved by requiring that the spaces A_k be chosen from a predetermined collection $\mathcal{A}$. An $\mathcal{A}$-cone decomposition of f is an ordinary cone decomposition in which all the spaces A_i are in the collection $\mathcal{A}$.

Now we can define the **cone length** of f to be

$$L(f) = \inf\{\mathrm{length}(\mathcal{D}) \,|\, \mathcal{D} \text{ is a cone decomposition of } f\}$$

(where, as usual, $\inf(\varnothing) = \infty$). The $\mathcal{A}$-cone length, $L_{\mathcal{A}}(f)$, is defined the same way, but $\mathcal{D}$ is required to run over just the $\mathcal{A}$-cone decompositions of f, not all cone decompositions.

One very important example is the collection of all wedges of spheres.

Exercise 8.60 Let X be a CW complex, and let $\mathcal{W}$ be the collection of all wedges of spheres. Show that $L_{\mathcal{W}}(X_n \hookrightarrow X) \leq \dim(X) - n$. Give an example to show that the inequality can be strict.

The **cone length** of a space is defined by $\mathrm{cl}(X) = L(* \to X)$ and, more generally, the $\mathcal{A}$-cone length of X is $\mathrm{cl}_{\mathcal{A}}(X) = L_{\mathcal{A}}(* \to X)$.

Exercise 8.61 Show that $\mathrm{cl}(X) = 0$ if and only if $X \simeq *$ and $\mathrm{cl}(X) = 1$ if and only if $X \not\simeq *$ and $X \simeq \Sigma Y$ for some Y.

Problem 8.62 Show that if f and g are homotopy equivalent maps, then $L_{\mathcal{A}}(f) = L_{\mathcal{A}}(g)$. Conclude that if $X \simeq Y$ then $\mathrm{cl}_{\mathcal{A}}(X) = \mathrm{cl}(Y)$.

Problem 8.63 Consider the homotopy pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\
C \arrow[r] & D.
\end{tikzcd}$$

Show that $L_{\mathcal{A}}(C \to D) \leq L_{\mathcal{A}}(A \to B)$.

Problem 8.64 Let Z be a space such that $\mathrm{map}_*(A, Z) \simeq *$ for all $A \in \mathcal{A}$, and let $f : X \to Y$.

(a) Show that if $\mathrm{cl}_{\mathcal{A}}(X) < \infty$ then $\mathrm{map}_*(X, Z) \simeq *$.

(b) What can you say about $\mathrm{map}_*(Y, Z)$ if $L_{\mathcal{A}}(f) < \infty$?

Decomposition of a Product. Next we consider the cone length of a product of two spaces. Suppose $\mathrm{cl}(X) = n$ and $\mathrm{cl}(Y) = m$; then what can we say about $\mathrm{cl}(X \times Y)$? Without loss of generality, we can assume that X has a system of subspaces $X_k \subseteq X$ such that

$$* \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n = X,$$

and each $X_{k+1} = X_k \cup CA_k$ (and similarly for Y). Then inside of $X \times Y$, we have the subspaces $X_i \times Y_j$, and more importantly, the subspaces

$$D_k = \bigcup_{i+j=k} X_i \times Y_j.$$

Notice that if the subspaces X_i and Y_j were the CW skeleta of X and Y, then D_k would be the k-skeleton of $X \times Y$ in the product CW structure. This suggests that the inclusion $D_k \hookrightarrow D_{k+1}$ might be a principal cofibration; this would imply the following formula for the cone length of a product.

Proposition 100 *For any two spaces X and Y,* $\mathrm{cl}(X \times Y) \leq \mathrm{cl}(X) + \mathrm{cl}(Y)$.

Problem 8.65

(a) Show that the inclusion $(X_i \times Y_{j-1}) \cup (X_{i-1} \times Y_j) \hookrightarrow X_i \times Y_j$ is a principal cofibration.

(b) Show that $D_k \hookrightarrow D_{k+1}$ is a principal cofibration.

(c) Prove Proposition 100.

Here is a challenge.

Exercise 8.66 Generalize Proposition 100 to formulas for $L(X \times Y)$, $\mathrm{cl}_{\mathcal{A}}(X \times Y)$ or $L_{\mathcal{A}}(X \times Y)$. You will have to impose certain closure conditions on $\mathcal{A}$, such as 'if $A, B \in \mathcal{A}$, then $A \vee B \in \mathcal{A}$.'

Exercise 8.67 Dualize the ideas of this section. Start by defining a principal fibration to be a map $f : X \to Y$ which fits into a fiber sequence $X \to Y \to B$. Define the fiber length of a map and a space, etc.

Lusternik-Schnirelmann Category. A subset $U \subseteq X$ is a **categorical subset** if the inclusion map $i : U \hookrightarrow X$ is homotopic to a constant map. A **Lusternik-Schnirelmann cover** (or **LS cover** for short) of X is a cover

$$X = U_0 \cup U_1 \cup \cdots \cup U_n$$

in which each set $U_k \subseteq K$ is a closed categorical subset X.[2] The least n for which X has such an LS cover is called the **Lusternik-Schnirelmann category**[3] of X (usually, simply the **category** of X); it is denoted $\mathrm{cat}(X)$.

Exercise 8.68 Show that $\mathrm{cat}(X) = 0$ if and only if $X \simeq *$ and that, $\mathrm{cat}(\Sigma X) \leq 1$ for any space X.

Proposition 101 *If X is a homotopy retract of Y, then* $\mathrm{cat}(X) \leq \mathrm{cat}(Y)$.

Problem 8.69

(a) Prove Proposition 101.

Hint If X is a homotopy retract of Y, then we have $X \xrightarrow{i} Y \xrightarrow{r} X$ with $r \circ i \simeq \mathrm{id}_X$. Show that if $U \subseteq Y$ is a categorical subset, then $i^{-1}(U) \subseteq X$ is also categorical.

(b) Show that if $X \simeq Y$, then $\mathrm{cat}(X) = \mathrm{cat}(Y)$.

Since Lusternik-Schnirelmann category is a homotopy invariant of spaces, it makes sense to study it using the machinery of homotopy theory. But the given definition is better suited to point-set topology than to homotopy theoretic analysis. So we reformulate the definition in terms of diagrams.

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[2] What we call an LS cover usually called **categorical cover**, but the word category is vastly overused, and I think there is no chance of confusion with the alternate name.

[3] There is no chance that this terminology will change!

EXERCISE 8.70 Consider the pointed space X, considered as the pair $(X, *)$. Then we can form the product pair $(X \times X, X \vee X)$, and more generally $(X, *)^k$ for $k \geq 1$. This is a pair $(X^k, T^k(X))$, where $T^k(X) \subseteq X^k$ is a certain subspace.

(a) Write down the set $T^k(X)$ explicitly.

(b) Suppose that X is cofibrant (so $* \to X$ is a cofibration). Show that $T^k(X) \hookrightarrow X$ is a cofibration, and determine its cofiber.

Proposition 102 (G.W. Whitehead) *If X is normal, then* $\mathrm{cat}(X) \leq n$ *if and only if there is a lift up to homotopy in the diagram*

$$\begin{tikzcd} & T^{n+1}(X) \arrow[d] \\ X \arrow[ur, dotted, "\lambda"] \arrow[r, "\Delta"] & X^{n+1} \end{tikzcd}$$

where Δ is the diagonal map.

PROBLEM 8.71

(a) Prove that if X is a normal space, then there any open cover $\{U_0, U_1, \cdots, U_n\}$ can be 'shrunk' to a cover $\{V_0, V_1, \cdots V_n\}$ with $V_k \subseteq \overline{V}_k \subseteq U_k$.

(b) Prove that in a normal space, if $C \subseteq U \subseteq X$, where C is closed and U is open, then there is a function $u : X \to I$ with $u|_C = 1$ and $u|_{X-U} = 0$.

(c) Suppose X is normal and $C \subseteq U \subseteq X$, where C is closed and U is open. Show that if a homotopy $C \times I \to Y$ extends to $U \times I \to Y$, then it extends to $X \times I \to X$. Conclude that if $C \hookrightarrow U$ is a cofibration, then $C \hookrightarrow X$ is a cofibration.

(d) Show that if X is normal, then X has a cover by closed sets $\{W_0, W_1, \ldots, W_n\}$ such that there are homotopies $H_k : X \times I \to X$ with $H_k(W_k \times 0) = *$.

(e) Prove Proposition 102.

HINT If λ exists, let $U_k = \{x \in X \mid \lambda(x)_{k+1} = *\}$.

Weak Category. There is a related notion, called **weak category**. First, let us define the **reduced diagonal** map $\overline{\Delta}_n : X \to X^{\wedge n}$ to be the composition

$$\begin{tikzcd} X \arrow[rr, "\overline{\Delta}_n"] \arrow[dr, "\Delta"'] & & X^{\wedge n} \\ & X^n, \arrow[ur, "q"'] & \end{tikzcd}$$

where $X^{\wedge n} = \overbrace{X \wedge X \wedge \cdots \wedge X}^{n}$. We say that $\mathrm{wcat}(X) \leq n$ if and only if $\overline{\Delta}_{n+1} \simeq *$.

PROBLEM 8.72

(a) Show that $\mathrm{wcat}(X) \leq \mathrm{cat}(X)$.

(b) Show that if $A \to X \to Y$ is a cofiber sequence, then

$$(\mathrm{wcat}(X) + 1) \leq (\mathrm{wcat}(A) + 1)(\mathrm{wcat}(Y) + 1).$$

PROBLEM 8.73 Suppose $f : A \to X$ be a map of connected CW complexes, and suppose that $\Sigma f : \Sigma A \to \Sigma X$ is a homotopy equivalence.

(a) Show that if Q is a connected CW complex, then $f \wedge \mathrm{id}_Q : A \wedge Q \to X \wedge Q$ is a homotopy equivalence.

HINT First prove it by induction for Q finite. For the infinite case, express Q as a colimit of its finite subcomplexes.

(b) Show that if $g : B \to Y$ is a map of connected CW complexes that suspends to a homotopy equivalence, then $f \wedge g$ is a homotopy equivalence.

(c) Show that $\mathrm{wcat}(A) \leq \mathrm{wcat}(X)$.

(d) Show that if $\Sigma X \simeq *$, then $\mathrm{wcat}(X) \leq 1$.[4]

EXERCISE 8.74 Carefully evaluate the extent to which the connectedness hypotheses in Problem 8.73 are necessary.

Category and Cone Length. It turns out that LS category is closely related to cone length.

PROBLEM 8.75

(a) Show that if $\mathrm{cat}(X) = n$, then X is a retract of a space $\overline{X}$ which is a union of $n + 1$ cones.

HINT If $X = U_0 \cup U_1 \cup \cdots \cup U_n$, then the inclusion $U_k \hookrightarrow X$ extends to a map $CU_k \hookrightarrow X$.

(b) Show that the space $\overline{X}$ that you found in part (a) is homotopy equivalent to $X \vee \Sigma W$, for some space W.

Proposition 103 *(Ganea) If X is a normal ANR,[5] then then* $\mathrm{cl}(X) \leq n$ *if and only if X has a closed cover $X = U_0 \cup U_1 \cup \cdots \cup U_n$ is a closed cover with each $U_k \simeq *$.*

PROBLEM 8.76 Prove Proposition 103.

Theorem 104

(a) $\mathrm{cat}(X) \leq n$ *iff X is a retract of a space with cone length $\leq n$.*

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[4] Such spaces do exist.

[5] Absolute Neighborhood Retract

(b) $\mathrm{cat}(X) \leq \mathrm{cl}(X) \leq \mathrm{cat}(X) + 1$

PROBLEM 8.77 Prove Theorem 104.

Proposition 105 *A normal space X is a coH space if and only if X is a retract of a suspension.*

EXERCISE 8.78 The dual result is true: if X is sufficiently nice, then X is an H space if and only if it is a retract of a loop space. Can you prove this fact by dualizing the coH space argument given here?

PROBLEM 8.79 Show that if X and Y are coH spaces, then $X \wedge Y$ is a cocommutative coH space.

The close relationship between cone length and category means that we can use our already-proved formula for the cone length of a product to derive a formula for the LS category of a product.

Theorem 106 *For any two spaces X and Y,* $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$.

PROBLEM 8.80 Prove Theorem 106.

8.11 Every Action has an Equal and Opposite Coaction

A map $f : X \to Y$ gives rise to long exact sequences sequences that are mostly exact sequences of (abelian) groups. But the terms at the end are only pointed sets, and the exactness is very weak. But it turns out that the the algebraic structure can be pushed back one more step, giving us some useful information about the injectivity of the map in question.

EXERCISE 8.81 Let $\mathcal{C}$ be an arbitrary pointed category, and suppose G is a grouplike object in $\mathcal{C}$. Define what it means for G to act on the object $X \in \mathcal{C}$. Then dualize to give a definition of a coaction of a cogroup object C on X.

Coactions in Cofiber Sequences. Let $f : A \to X$ and consider maps from the mapping cone C_f to some other space Y. Since C_f is defined as a pushout, the map u is determined by the diagram

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{in}_0"'] \arrow[dr, phantom, "\text{pushout}"] & X \arrow[d] \arrow[ddr, bend left, "v"] & \\
CA \arrow[r] \arrow[drr, bend right, "\overline{H}"'] & C_f \arrow[dr, "u"] & \\
& & Y.
\end{tikzcd}$$

Thus the map $u : X \cup CA \to Y$ is equivalent to two pieces of information: a map $v : X \to Y$, and a homotopy $H : v \circ f \simeq *$.

Exercise 8.82 Suppose we are given two homotopies $H, K : v \circ f \simeq *$. Show that if they are homotopic homotopies in the sense of Section **??**, then the corresponding maps $u_H : X \cup CA \to Y$ and $u_K : X \cup CA \to Y$ are homotopic.

Now consider the inclusion $\mathrm{in}_{\frac{1}{2}} : A \to CA$ given by $a \mapsto [a, \frac{1}{2}]$. This is a cofibration, and the cofiber is homeomorphic (by parameter changes) to $CA \vee \Sigma A$. Attaching this to X gives

$$\alpha : X \cup CA \to (X \cup CA) \vee \Sigma A.$$

Exercise 8.83 Write out the details of 'by parameter changes' and 'Attaching this to X'

Problem 8.84 Show that the map α defines a coaction of ΣA on $X \cup CA$ in the category $\mathrm{H}\mathcal{T}_*$.

The Action of $[\Sigma A, Y]$ on $[C_f, Y]$. If we apply a functor of the form $[\,?\,, Y]$, the map α_* may be naturally identified with

$$\alpha_* : [X \cup CA, Y] \times [\Sigma A, Y] \to [X \cup CA, Y].$$

Note that, although $[X \cup CA, Y]$ is only a pointed set (as far as we know), $[\Sigma A, Y]$ is a naturally a group.

Problem 8.85 Show that the map α_* is an action of the group $[\Sigma A, Y]$ on the set $[X \cup CA, Y]$.

We will use exponential notation for this action: $\alpha_*(u, \delta)$ will generally be written u^δ. This action is natural in two ways.

Problem 8.86

(a) Suppose $f : Y \to Z$. Show that $f_*(u^\delta) = (f_*(u))^{f_*(\delta)}$.

(b) Suppose the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\
B \arrow[r] & Y
\end{tikzcd}$$

commutes up to homotopy, with induced map $h : X \cup CA \to Y \cup CB$. Show that $h^*(u^\delta) = (h^*(u))^{(\Sigma f)^*(\delta)}$.

The most important property of this action, however, is that it gives us a useful exactness property in the pointed set end of the long cofiber sequence.

Proposition 107 *In the sequence*

$$[X, Y] \xleftarrow{j^*} [X \cup CA, Y] \xleftarrow{\partial^*} [\Sigma X, Y]$$

$j^*(u) = j^*(v)$ *if and only if there is* $\delta \in [\Sigma A, Y]$ *such that* $u = v^\delta$.

PROBLEM 8.87 Let $A \xrightarrow{f} X \xrightarrow{j} X \cup CA \xrightarrow{\partial} \Sigma A \longrightarrow \cdots$ be the beginning of the long cofiber sequence of f, and consider the action of $[\Sigma A, Y]$ on $[X \cup CA, Y]$.

(a) Show that if $u = v^\delta$, then $j^*(u) = j^*(v)$.

(b) Show that if $j^*(u) = j^*(v)$, then u and v can be replaced with homotopic maps such that $j \circ u = j \circ v$. Call this common map $w : X \to Y$. Then u is built from the map w and a certain homotopy $H_u : w \circ f \simeq *$ and v is built from the map w and another homotopy $H_v : u \circ f \simeq *$.

(c) Show that H_v is homotopic to the homotopy $H_u + (\overleftarrow{H}_u + H_v)$.

(d) Prove Proposition 107.

Find example where only the trivial map becomes trivial, but lots of other maps are identified.

PROBLEM 8.88 Show that the map $[X, Y] \to \langle X, Y \rangle$ is a bijective if Y is simply-connected.

HINT Look at Problem **??**

A Diagram Lemma. Suppose we are given the (homotopy) commutative square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl, dotted] \\
C \arrow[r] & D.
\end{tikzcd}$$

It sometimes happens that we want to find a dotted arrow that will make both triangles commute up to homotopy.

Exercise 8.89 Give an example of a square in which no such map exists.

Hint Take for granted that there are noncontractible spaces.

Even though these maps do not generally exist, there are many special situations in which they can be found.

Problem 8.90 Let $f : X \to Y$. Show that the following are equivalent:

1. The map $\Omega f : \Omega X \to \Omega Y$ has a homotopy section $g : \Omega Y \to \Omega X$.
2. For every space Z, the induced map $f_* : [Z, X] \to [Z, Y]$ is surjective.

Proposition 108 *Suppose that there is a map $\alpha : Z \to A$ such that the top row in the diagram*

$$\begin{tikzcd}
Z \arrow[r, "\alpha"] & A \arrow[r, "j"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, dotted, "\xi"'] \\
 & C \arrow[r, "f"] & D.
\end{tikzcd}$$

is a cofiber sequence, and suppose also that the map $f : C \to D$ satisfies the conditions of Problem 8.90. Then the dotted arrow $\xi : B \to C$ can be found so that both triangles commute up to homotopy if and only if the composite $Z \to A \to C$ is trivial.

One implication is trivial.

Exercise 8.91 Show that if ξ exists, then $Z \to A \to C$ must be trivial.

Now let's work on the real substance of the proof.

Problem 8.92 Now suppose that the composite $Z \to A \to C$ is trivial.

- (a) Show that there is a map $\zeta : B \to C$ making the upper left triangle commute up to homotopy. Describe the set of all such maps.
- (b) Show that there is a $\delta \in [\Sigma Z, D]$ such that $h \simeq (f \circ \zeta)^\delta$.
- (c) Finish the proof of Proposition 108.

Action of ΩY on F. The discussion above is purely formal, and hence it is easily dualized.

Exercise 8.93 Dualize the discussion above.

Having left the dualization to you, we can develop the action of ΩY on F in a different way. When we introduced fibrations, we gave an alternate characterization in terms of lifting functions. Given a map $p : E \to B$, form the pullback square

$$\begin{tikzcd}
E \times_B B^I \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
B^I \arrow[r, "@_0"] & B.
\end{tikzcd}$$

Since $p \circ @_0 = @_0 \circ p_* : E^I \to B$, there is a uniquely defined map

$$q = (@_0, p_*) : E^I \to E \times_B B^I.$$

You proved in Problem **??** that p is a fibration if and only it has a lifting function – i.e., a section of the map q.

Suppose now that $p : E \to B$ is a fibration, so that we have a lifting function $\lambda : E \times_B B^I \to E^I$. This lifting function is a rule assigning to each path $\omega : I \to B$ and each point $e \in E$ such that $p(e) = \omega(0)$ a path $\widehat{\omega} : I \to E$ such that $\widehat{\omega}(0) = e$ and $p \circ \widehat{\omega} = \omega$.

What if we focus on the paths from $*$ to $*$ in B? Then we obtain a map μ by restriction of the domain and target in the diagram

$$\begin{tikzcd}
F \times \Omega Y \arrow[rr, "\mu"] \arrow[d] & & F \arrow[d] \\
E \times_Y Y^I \arrow[r, "\lambda"] & E^I \arrow[r, "@_1"] & E.
\end{tikzcd}$$

Problem 8.94 Show that the homotopy class of μ is independent of the choice of lifting fuction λ.

Exercise 8.95 Show that the map $F \times \Omega Y \to F$ you get by dualizing the cofiber case is the same as the map you get using the lifting function approach.

The map μ induces $\mu_* : [A, F \times \Omega B] \to [A, F]$, which may be naturally identified with

$$\mu_* : [A, F] \times [A, \Omega Y] \to [A, F].$$

We again use exponential notation, so that if $u \in [A, F]$ and $\delta \in [A, \Omega Y]$, then $\mu_*(u, \delta) = u^\delta$.

Problem 8.96 Show that μ_* defines an action of $[A, \Omega B]$ on $[A, F]$. Clearly write out the ways in which this action is natural.

Proposition 109 *In the sequence*

$$[A, \Omega Y] \xrightarrow{\partial_*} [A, F] \xrightarrow{i_*} [A, X],$$

$i_*(u) = i_*(v)$ *if and only if there is a* $\delta \in [A, \Omega Y]$ *such that* $u = v^\delta$.

Problem 8.97 Prove Proposition **??**.

8.12 Prelude to Phantom Maps

CW complexes are easy to work with because of their step by step construction. Since each skeleton is related to the next one by a cofiber sequence, we can apply our exact sequences and coactions to understand the maps out of the skeleta of X. But what about X itself? If X is finite-dimensional, then X is one of its skeleta, and we can get understand $[X, Y]$ by this method. But if X is infinite-dimensional, then after the skeleta have been constructed, and the mapping sets $[X_n, Y]$ have been determined, we still have the problem of relating those sets to the X, which is the (homotopy) colimit of the skeleta.

This problem can be generalized a bit. Let X be the colimit of the telescope diagram

$$X_1 \to X_2 \to \cdots \to X_n \to X_{n+1} \to \cdots$$

in a category $\mathcal{C}$. Then for any $Y \in \mathcal{C}$, the maps $X \to \mathcal{C}$ are completely determined by the compositions $X_n \to X \to Y$. But what about the homotopy classes of maps? Suppose $f, g : X \to Y$ and $f|_{X_n} \simeq g|_{X_n}$ for each n, does it follow that $f \simeq g$?

Of particular importance is the case in which the telescope diagram is the diagram of CW skeleta of X and $g = *$. We say that $f : X \to Y$ is a **phantom map** if $f|_{X_n} \simeq *$ for all n. Write $\mathrm{Ph}(X, Y) \subseteq [X, Y]$ for the set of all phantom maps from X to Y. Clearly the trivial map is a phantom map, but are there nontrivial phantom maps? We have not yet established the basic computations needed to answer this question, but we are ready to set up a good deal of the basic theory of phantom maps in preparation for later attacks on the problem.

Maps Out of a Telescope. Let X be the homotopy colimit of the telescope diagram

$$X_1 \to X_2 \to \cdots \to X_n \to X_{n+1} \to \cdots$$

of spaces. Because maps from homotopy colimits are not uniquely determined by the diagram, we first show that the condition $f|_{X_n} \simeq g|_{X_n}$ for each n actually makes sense.

EXERCISE 8.98 Let $f, g : X \to Y$.

(a) Show that there are maps $j_n : X_n \to X$ induced from the diagram, and that they are unique up to homotopy equivalence of maps.

HINT Induce them from maps of diagrams.

(b) Suppose j_n and $\widetilde{j}_n$ are two such induced maps. Show that $f \circ j_n \simeq g \circ j_n$ if and only if $f \circ \widetilde{j}_n \simeq g \circ \widetilde{j}_n$.

This shows that the question is well-defined in that it does not depend on which choice of map $j_n : X_n \to X$ we choose.

Universal Phantom Maps. Choose your favorite induced maps $j_n : X_n \to X$ and define

$$j = (j_n) : \bigvee_1^\infty X_n \to X.$$

To simplify notation, we write $W_k = \bigvee_1^k X_n$ and $W = \bigvee_1^\infty X_n$. With this notation, j is a map $W \to X$.

Problem 8.99

(a) Show that $f \circ j_n \simeq g \circ j_n$ for all n if and only if $f \circ j \simeq g \circ j$.

(b) Show that f is a phantom map if and only if there is a homotopy factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\Theta_X"'] & & Y \\ & C_j \arrow[ur, dotted] & \end{tikzcd}$$

where $\Theta_X : X \to C_j$ is the cofiber of the map j.

(c) Show that Θ_X is a phantom map. Conclude that there are nontrivial phantom maps out of X if and only if $\Theta_X \not\simeq *$.

Because of Problem 8.99, the map $\Theta_X : X \to C_j$ is called the **universal phantom map** out of X.[6] Let us study this map in more detail. The **shift map** is the map SHIFT : $W \to W$ given by

$$\begin{tikzcd} W_n \arrow[r, "i_n"] \arrow[d, hook] & W_{n+1} \arrow[d, hook] \\ W \arrow[r, "\text{SHIFT}"] & W. \end{tikzcd}$$

Problem 8.100

(a) Show that there is a homotopy pushout square

$$\begin{tikzcd} W \vee W \arrow[r, "\nabla"] \arrow[d, "(\text{SHIFT}, \mathrm{id}_W)"'] & W \arrow[d] \\ W \arrow[r, "j"] & X. \end{tikzcd}$$

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[6] Mention terminology ‘versal.’ Get reference from H. Miller.

(b) Show that $C_j \simeq \Sigma W$. Conclude that there is a cofiber sequence

$$W \xrightarrow{j} X \xrightarrow{\Theta_X} \Sigma W \xrightarrow{q} \Sigma W \xrightarrow{\Sigma j} \Sigma X \longrightarrow \cdots .$$

Here is a more hands-on approach to identifying the spaces and maps in the long cofiber sequence generated by j. Define maps $\Sigma X_n \to C_j$ by the pushout property in the diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & CX_n \arrow[d] \arrow[dr, bend left] & \\
CX_n \arrow[r] \arrow[d] & \Sigma X_n \arrow[dr, dotted, "\xi_n"] & CW \arrow[d] \\
CX_{n+1} \arrow[r] & CW \arrow[r] & C_j .
\end{tikzcd}$$

Problem 8.101 Show that $\xi = (\xi_n) : \Sigma W \to C_j$ is a homotopy equivalence.

Then since $X \to C_j$ is a cofibration, we can take q to be the map which pinches X to a point.

Problem 8.102 Make the identification $[\Sigma W, Y] = [\bigvee \Sigma X_n, Y] \cong \prod [\Sigma X_n, Y]$.

(a) Determine the induced map $q^* : \prod [\Sigma X_n, Y] \to \prod [\Sigma X_n, Y]$.

Hint Since the domain is ΣW, it suffices to determine the restrictions $\Sigma X_n \to W$ for each n. Follow the cones!

(b) Determine the action of $\prod [\Sigma X_n, Y]$ on $\prod [\Sigma X_n, Y]$.

Problem 8.103 Show that if $\Theta_X \simeq *$ then Σj has a section up to homotopy.

Inverse Limits and $\lim^1$ for Groups. We will find a formula for $\mathrm{Ph}(X, Y)$ in terms of the spaces X_n and Y. This formula will involve certain group-theoretic functors which we will define and study.

Now homological algebra makes its appearance. First we give an explicit construction for the limit of a tower of sets. Given

$$A_1 \xleftarrow{p_2} A_2 \xleftarrow{p_3} A_3 \longleftarrow \cdots$$

of groups, we form the product $\prod A_n$ and define the shift map

$$\textsc{shift} : \prod A_n \to \prod A_n$$

by the formula $\textsc{shift}(a_1, a_2, \ldots) = (p_2(a_2), p_3(a_3), \ldots)$.

Problem 8.104

(a) Let
$$L = \{(a_1, a_2, \ldots) \,|\, p_n(a_n) = a_{n-1} \quad \text{for all } n\}.$$
Show that L, with the obvious maps $L \to A_n$, is a limit for the tower.

(b) Show that if the tower is a tower of groups and homomorphisms, then L is also a group, and in fact it is the limit of the tower.

(c) Show that if it is a tower of abelian groups, then $L = \ker(\mathrm{id}_{\prod A_n} - \textsc{shift})$.

Our next functor is most naturally defined for towers of abelian groups. In this case, the tower gives rise to the map

$$\sigma = \big(\mathrm{id}_{\prod A_n} - \textsc{shift}\big) : \prod A_N \to \prod A_n,$$

and from there to the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker(\sigma) \arrow[r] & \prod A_n \arrow[r, "\sigma"] & \prod A_n \arrow[r] & \mathrm{coker}(\sigma) \arrow[r] & 0. \end{tikzcd}$$

We have already identified the kernel of σ: it is the limit of the tower. The cokernel is also important, and it is known as $\lim^1$ of the tower. The usual practice is to omit the maps in the tower from the notation, so that we write $\lim A_n$ and $\lim^1 A_n$ for these groups, even though they depend crucially on the maps involved.

When the groups are not abelian, we can still define $\lim^1 A_n$. First define a action of $\prod A_n$ on itself by the rule

$$(a_1, \ldots, a_n, \ldots)^{(b_1, \ldots, b_n, \ldots)} = (b_1^{-1} a_1 p_2(b_2), \ldots, b_n^{-1} a_n p_{n+1}(b_{n+1}), \ldots).$$

Then $\lim^1 A_n$ is the orbit space

$$\lim{}^1 A_n = \Big(\prod A_n\Big) / (\text{action}).$$

Exercise 8.105 Show that if the groups A_n are abelian then the two definitions of $\lim^1 A_n$ agree.

Determining the Set of Phantom Maps. We wish to find a formula for this set in terms of the spaces X_n and Y. We will place $\mathrm{Ph}(X, Y)$ in a short exact sequence involving lim and $\lim^1$.

Problem 8.106

(a) Suppose that all the maps in the telescope diagram $X_1 \xrightarrow{i_1} X_2 \xrightarrow{i_2} \cdots$ are cofibrations. Let $\{f_n : X_n \to Y\}$ be a sequence of maps such that $f_{n+1} \circ i_n \simeq f_n$. Show that there are maps $\{\phi_n : X_n \to Y\}$ such that $\phi_n \simeq f_n$ and $\phi_{n+1} \circ i_n = \phi_n$.

(b) Show that the map

$$[X, Y] \to \prod [X_n, Y]$$

given by $f \mapsto (f \circ j_n)$ is surjective.

Theorem 110 *Let X be the colimit of the telescope diagram $X_1 \to X_2 \to \cdots$. Then for any space Y, there is a short exact sequence*

$$* \longrightarrow \mathrm{Ph}(X, Y) \longrightarrow [X, Y] \longrightarrow \lim [X_n, Y] \longrightarrow *$$

of pointed sets.

PROBLEM 8.107 Show that $\mathrm{Ph}(X, Y) \cong \lim^1 [\Sigma X_n, Y]$. Rewrite the exact sequence of Theorem 110 using $\lim^1$.

Part II

Four Topological Inputs

Introduction to Part 3

What we have done so far has essentially been establishing that fibrations, cofibrations, pushouts, etc. satisfy certain formal properties, and then exploiting those properties. Thus, the bulk of what we have done so far is applicable in just about any category where homotopy theory can be defined.

But the category of topological spaces has more to it: we actually know what the objects are, and we can work with elements and open sets, if we have to, to prove things about them. It would be silly to throw away all of these tools when they are available. And furthermore, since the theory so far is purely formal, it is consistent with the possibility that all spaces are contractible! (We have asserted that $S^n \not\simeq *$, but not yet proved it.) There are four topological inputs that make classical homotopy theory tick.

The first is more of a philosophy than a theorem: we want to study CW complexes. This is justified philosophically rather than mathematically: the spaces that we generally think of are CW complexes (especially manifolds); many interesting problems of algebra or analysis can be reduced to homotopy theoretical questions about CW complexes; and so forth. Since we are deciding to concentrate on CW complexes, we get some extra information with with to study spaces: the concept of dimension. We develop these ideas in Chapter 9.

Our second topological input is the Cellular Approximation Theorem, which says that a map $f : X \to Y$ of CW complexes is homotopic to a **cellular map**, i.e., a map that respects dimension. This boils down to a simple fact of linear algebra (or measure theory): a union of finitely many hyperplanes in $\mathbb{R}^n$ is a proper subset of $\mathbb{R}^n$.

The third input is a theorem of Hurewicz: a map that is locally a fibration is actually a fibration. This takes the homotopy-theoretical form of the First Cube Theorem, which mixes homotopy pushout squares with homotopy pullback squares.

The fourth and final topological input is a simple lemma: suppose $B \to$

$D \leftarrow C$ is a prepushout, $B \to D$ is a cofibration and $C \to D$ is a fibration. Then form the pullback A, and get maps $A \to B$ and $A \to C$. We know that $A \to B$ is a fibration, formally. Amazingly, in the category of topological spaces, $A \to C$ is a cofibration! The Second Cube Theorem is the homotopy-theoretical instantiation of this easily proved but very surprising lemma.

Chapter 9

Dimension in Homotopy Theory

This chapter is unlike the others in this Part, in that we do not have a single, well-defined theorem that we wish to establish. Rather, we explore the ways in which the concept of dimension can be used in the study of homotopy theory.

In the introduction to Part 3, we argued that we are fundamentally interested in the homotopy theory of CW complexes. CW complexes are not simply topological spaces: they have extra structure conferred upon them by virtue of their their step-by-step construction. They are filtered by their skeleta, and it makes sense to talk about the dimension of a CW complex.

We have seen in Problem **??** that in order to decide whether or not a map $f : X \to Y$ is a homotopy equivalence, it is sufficient to show that for every space K, the induced map $f_* : [K, X] \to [K, Y]$ is bijective. But we may not be able to check this condition for all spaces K; perhaps we can only check it for CW complexes, or for CW complexes of dimension at most n. This leads to the concept of *n-equivalence*.

Analogously, we can tell whether X is contractible by checking whether $[K, X] = *$ for all K or not. If $[K, X] = *$ for all CW complexes, we say that X is *weakly contractible*; and if we can only check this for CW complexes of dimension at most n, then we arrive at the notion of an *n-connected space*.

In this chapter we explore the relation between n-equivalence and n-connectivity. We establish 5 different reformulations of the concept of n-equivalence, and use them to prove the celebrated J.H.C. Whitehead theorem, which says that a map which induces bijections $f_* : \pi_k(X) \to \pi_k(Y)$

for all k induces bijections $f_* : [K, X] \to [K, Y]$ for *all* CW complexes K.

9.1 Induction Principles for CW Complexes

The step-by-step construction of CW complexes makes it possible to prove things about them by induction on their skeleta. This section contains some technical results that facilitate this kind of argument.

Attaching One More Cell. If X is a CW complex and $K \subseteq X$ is a subcomplex, then it may happen that $K_0 = X_0$; that is, that the 0-skeletons are the same. If so, then it may or may not happen that $X_1 = K_1$, and so on. If K is a *proper* subcomplex, then there must be some n such that $K_n \neq X_n$. If n is the smallest dimension for which $X_n \neq K_n$, then we can attach to K one of the n-cells that it does not contain. This proves the following.

Lemma 111 *If X is a CW complex and K is a proper subcomplex, then there is another subcomplex L such that*

(a) $K \subseteq L \subseteq X$, *and*

(b) $L = K \cup_\lambda D^n$ *for some map* $\lambda : S^{n-1} \to K$.

Many of the proofs in this chapter will make essential use of Lemma 111, in the following way. We wish to show that some property is true of X, so we let $K \subseteq X$ be a subcomplex which is maximal with that property[1] – we hope to show that $K = X$. So we assume that K is a proper subcomplex of X, and find a slightly larger subcomplex L as in the lemma. Using the close connection between K and L, we then prove that the property holds for L, contradicting the maximality of K, and thereby proving that X has the desired property.

Composing Infinitely Many Homotopies. Our second induction principle addresses the following question: suppose that for each n you have a homotopy $H_n : f|_{X_n} \simeq g|_{X_n}$. As we have seen in Section 8.12, it is not at all obvious whether or not f and g are homotopic. However, if we have a bit more control over the homotopies $f_n \simeq g_n$, we *can* piece them together to obtain a homotopy $H : f \simeq g$.

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[1] How do we know there is such a subcomplex? The main tool for proving the existence of maximal gadgets is Zorn's Lemma.

Proposition 112 *Let $f, g : X \to Y$, where X is a CW complex. Suppose there is an infinite sequence of maps $f_n : X \to Y$ (for $n \geq 1$) and homotopies $H_n : f_n \simeq f_{n+1}$ which satisfy the following conditions:*

1. *$f \simeq f_0$,*
2. *for each n, $f_n|_{X_n} = g|_{X_n}$,*
3. *$H_n : f_n \simeq f_{n+1}$ is a homotopy that is constant on X_n.*

Then $f \simeq g$.

EXERCISE 9.1 Show that if X is a CW complex and $J : X \times I \to Y$ is a function, then J is continuous if and only if $J|_{X_n \times I} : X_n \times I \to Y$ is continuous for each n.

PROBLEM 9.2 Let $H : f \simeq f_0$, and reparametrize it so that it takes place from $t = 0$ to $t = \frac{1}{2}$; call the reparametrized homotopy J_0. More generally, reparametrize H_n so that it takes place on the interval $[1-\frac{1}{n+2}, 1-\frac{1}{n+3}]$ and call the reparametrized homotopy J_n.

(a) Show that the homotopies J_n for $n \geq 0$ glue together to give a continuous function $\widetilde{J} : X \times [0, 1) \to Y$.

(b) Show that $\widetilde{J}$ can be extended to a homotopy $J : X \times I \to Y$, and prove Proposition 112.

9.2 n-Equivalences and Connectivity of Spaces

In this section, we introduce the system of weaker notions of homotopy equivalence, called n-equivalences, and a corresponding collection of measures of the triviality of spaces, called n-connectivity.

n-Equivalences. A map $f : X \to Y$ in $\mathcal{T}_*$ is an **n-equivalence** if the induced map

$$f_* : [K, X] \to [K, Y]$$

is an isomorphism for every CW complex K with $\dim(K) < n$ and is a surjection if $\dim(K) \leq n$. We say that f is an **∞-equivalence** if it is an n-equivalence for each n. Such a map is also called a **weak homotopy equivalence** or simply a **weak equivalence**.

EXERCISE 9.3 Show that if f and g are homotopy equivalent maps, then f is an n-equivalence if and only if g is an n-equivalence.

One could also define n-equivalences of unpointed spaces. But we work only with the pointed version, because an n-equivalence in $\mathcal{T}_*$ is automatically an n-equivalence in $\mathcal{T}_\circ$.

Exercise 9.4 Prove that an n-equivalence in $\mathcal{T}_*$ is automatically an n-equivalence in $\mathcal{T}_\circ$. Is the converse true?

Problem 9.5

(a) Show that a pointed n-equivalence is automatically an unpointed n-equivalence, without any conditions of the spaces involved.

(b) Let $F : \mathcal{T}_* \to \mathcal{T}_\circ$ be the forgetful functor, and let $f : X \to Y$ in $\mathcal{T}_*$, where Y is simply connected and X is cofibrant. Show that if f is a pointed n-equivalence if and only if $F(f)$ is an unpointed n-equivalence.

Hint Use Problem 8.88.

What kind of map does an n-equivalence induce on homotopy groups?

Problem 9.6 Show that if f is an n-equivalence, then the induced map $f_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$, no matter what basepoint is chosen.

Problem 9.7 Suppose $f : X \to Y$ is an n-equivalence. What can you say about Ωf? What can you say about $f_* : \mathrm{map}(A, X) \to \mathrm{map}(A, Y)$?

Next we consider composition of n-equivalences.

Problem 9.8 Let $f : X \to Y$ and $g : Y \to Z$. Suppose you know that two of the maps f, g and $g \circ f$ are n-equivalences (with $n \leq \infty$). What can you say about the third map?

Connectivity of Spaces. A space $X \in \mathcal{T}_*$ is **n-connected** if $[K, X] = *$ for all CW complexes K with $\dim(K) \leq n$, and X is ∞-connected if $[K, X] = *$ for all finite dimensional CW complexes K.

Exercise 9.9 Show that the pointed and unpointed notions of n-connected are equivalent.

Problem 9.10

(a) Show that 0-connected is synonymous with path connected.

(b) Show that a space is 1-connected if and only if it is simply connected.

The **connectivity** of a space X is the greatest n for which X is n-connected; we'll use the notation $\mathrm{conn}(X) = n$ to mean that X is n-connected but not $(n+1)$-connected. This is a homotopy invariant of spaces that measures how close X is to being the trivial space $*$.

Problem 9.11 Let $F \to E \to B$ be a fibration sequence.

(a) Suppose two of the three spaces are n-connected. What can you say about the third?

(b) Suppose X is n-connected and Y is m-connected. What is the connectivity of $X \times Y$?

(c) How do the connectivities of X and ΩX compare?

The connectivity of a space can be determined by studying its homotopy groups.

PROBLEM 9.12 Let X be path-connected, and show the following are equivalent:

1. $\pi_k(X) = *$ for all $k \leq n$
2. X is n-connected.

HINT Work by induction on a CW decomposition of K; what is the suspension of $\coprod S^{k-1}$?.

PROBLEM 9.13 Show that the following are equivalent:

1. X is n-connected
2. $X \to *$ is an $(n+1)$-equivalence
3. $* \to X$ is an n-equivalence

Food For Thought. Our results so far bring up some interesting questions.

1. Suppose f is an n-equivalence. What can be said about the map Σf?
2. Suppose $A \to B \to C$ is a cofiber sequence and two of the three spaces are n-connected. What can be said about the third?
3. Let $f : X \to Y$. Can we decide whether or not f is an n-equivalence from knowledge of the cofiber? How are the connectivities of the fiber and the cofiber of f related to one another?

These questions cannot be answered with our currently available tools, because connectivity is defined as a target-type concept, and we are asking here about how it behaves with respect to domain-type constructions. This kind of question cannot be answered based on the formalities of the theory; it requires knowledge of the special features of the category of *topological spaces.*

9.3 Reformulations of n-Equivalences

In this section, we establish alternate characterizations of n-equivalences. These are extremely important, because they make it possible for us to use n-equivalences to *prove things*. Specifically, parts (b) and (d) are useful when you want to verify that a given map is an n-equivalence, and parts (c) and (e) are useful when you are proving a statement using CW induction.

Let $f : X \to Y$ be a map in $\mathcal{T}_\circ$. Then for each point $x \in X$ we obtain a pointed map $f_x : (X, x) \to (Y, f(x))$.

Theorem 113 *Let $f : X \to Y$ be a map such that $\pi_0(f)$ is onto, and let $n \leq \infty$. The following are equivalent:*

- *(a) f is an n-equivalence*
- *(b) for every $x \in X$, the induced map $(f_x)_* : \pi_k(X, x) \to \pi_k(Y, f(x))$ is an isomorphism for $k < n$ and a surjection for $k = n$*
- *(c) in any strictly commutative diagram of the form*

$$\begin{tikzcd}
S^k \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & D^{k+1} \arrow[dl, dotted, "\beta_1"'] \arrow[dd, "\mathrm{in}_1"] \\
& X \arrow[dd, "f"] & \\
(D^{k+1} \times 0) \cup (S^k \times I) \arrow[rr, hook, crossing over] \arrow[dr, "\beta_0 \cup H_{S^k}"'] & & D^{k+1} \times I \arrow[dl, dotted, "H_{D^{k+1}}"] \\
& Y &
\end{tikzcd}$$

  with $k < n$, the dotted arrows can be filled in to make the whole diagram strictly commutative.

- *(d) in any strictly commutative diagram of the form*

$$\begin{tikzcd}
S^k \arrow[r, "\alpha_1"] \arrow[d, hook] & X \arrow[d, "f"] \\
D^{k+1} \arrow[r, "\beta_0"'] \arrow[ur, dotted, "\beta_1"] & Y
\end{tikzcd}$$

  with $k < n$, the dotted arrow can be filled in so that the upper triangle commutes on the nose, and the lower triangle commutes up to a homotopy which is constant on S^k.

(e) *if A is any space, and B is obtained from A by attaching cells of dimension at most n.*[2] *then in the strictly commutative diagram*

$$\begin{tikzcd}
A \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & B \arrow[dl, dotted, "\beta_1"'] \arrow[dd, "\mathrm{in}_1"] \\
& X \arrow[dd, "f"] & \\
(B \times 0) \cup (A \times I) \arrow[rr, hook] \arrow[dr, "\beta_0 \cup H_A"'] & & B \times I \arrow[dl, dotted, "H_B"] \\
& Y &
\end{tikzcd}$$

the dotted arrows can be filled in so that the entire diagram commutes.

Theorem 113(e) is known as the **Homotopy Extension Lifting Property**, often abbreviated HELP. The proof of Theorem 113 is quite involved, and it is given, in parts, in the next section.

Corollary 114 *If $f : X \to Y$ is a map of connected CW complexes, then f is an n-equivalence if and only if the homotopy fiber F is $(n-1)$-connected.*

PROBLEM 9.14 Prove Corollary 114.

Before proceeding to the proof of Theorem 113, let's look at an interesting implication which shows how knowledge of homotopy groups can lead to substantial conclusions.

PROBLEM 9.15 Let $f : X \to Y$ such that $f_* : \pi_k(X) \to \pi_k(Y)$ is an isomorphism for $k \leq n$ and Y is path connected.

(a) Suppose Y is a CW complex with $\dim(Y) \leq n$. Show that Y is a retract, up to homotopy, of X.

(b) Show that if both X and Y are CW complexes of dimension $< n$, then f is a homotopy equivalence.

PROBLEM 9.16 Let $f : X \to Y$ be a map of path connected spaces. Show that f is an n-equivalence if and only if the homotopy fiber F is $(n-1)$-connected.

In particular, this shows that f is a weak equivalence if and only if the fiber F is weakly contractible. But what about the cofiber?

PROBLEM 9.17 Suppose $f : X \to Y$ and $C_f \simeq *$. Show that $\Sigma f : \Sigma X \to \Sigma Y$ is a homotopy equivalence.

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[2] Thus B is a *relative* CW complex with dimension at most n.

Sometimes an n-equivalence is referred to as an n-connected map. The connectivity of a map f is the greatest n for which f is n-connected.

9.4 Proof of Theorem 113

In this section we prove Theorem 113. We begin by reducing the proof to a conceptually simpler special case: f is the inclusion of a subspace, and $n < \infty$. Next we prove the equivalence of parts (b), (c) and (d); these proofs depend on certain decompositions of spheres and disks as unions of disks and cylinders. Finally, we show that parts (a), (b), and (e) are equivalent; the proof is grounded in CW induction.

9.4.1 Reductions

Before attacking the proof of Theorem 113, we show that certain conceptual simplifications can be made.

Problem 9.18

(a) Suppose f and g are homotopy equivalent maps. Show that if f satisfies one of the conditions (a), (b), (c), (d) or (e) of Theorem 113, then so does g.

(b) Show that it suffices to prove Theorem 113 in the case $f : X \to Y$ is the inclusion of a subspace.

Hint Use the mapping cylinder.

(c) Show that it suffices to prove Theorem 113 under the assumption that X and Y are both path connected.

(d) Show that it suffices to prove Theorem 113 in the case $n < \infty$.

9.4.2 Equivalence of Parts (b), (c) and (d) of Theorem 113

To prove that Theorem 113(b) implies Theorem 113(c) and Theorem 113(c) implies Theorem 113(d), we have to be able recognize the inclusion of a sphere into a disk when we see it.

Problem 9.19

(a) Show there is a homeomorphism $u : D^{k+1} \to (D^{k+1} \times 0) \cup (S^k \times I)$ making the diagram

$$\begin{tikzcd}
S^k \arrow[r, "\mathrm{id}"] \arrow[d, "i"'] & S^k \arrow[d, "\mathrm{in}_1"] \\
D^{k+1} \arrow[r, "u"] & (D^{k+1} \times 0) \cup (S^k \times I),
\end{tikzcd}$$

strictly commutative,

(b) Show that there is a homeomorphism $v : D^{k+2} \to D^{k+1} \times I$ making the diagram

$$\begin{tikzcd}
S^{k+1} \arrow[r] \arrow[d] & (D^{k+1} \times 0) \cup (S^k \times I) \cup (D^{k+1} \times 1) \arrow[d] \\
D^{k+2} \arrow[r, "v"] & D^k \times I
\end{tikzcd}$$

strictly commutative.

Now show that Theorem 113(b) implies Theorem 113(c).

Problem 9.20 Take Theorem 113(b) as known.

(a) Show that α_1 extends to a map $\xi : D^{k+1} \to X$.

(b) Explain how the maps β_0, H_{S^k} and ξ define a map $Q : S^{k+1} \to Y$.

(c) Show that there is a map $R : S^{k+1} \to X$ such that $f \circ R \simeq Q$.

(d) Let $\beta_1 : D^{k+1} \to X$ be the composition

$$\begin{tikzcd}
D^{k+1} \arrow[rr, "\beta_1"] \arrow[dr, "c"'] & & Y \\
& D^{k+1} \vee S^{k+1}, \arrow[ur, "(\xi{,}-R)"'] &
\end{tikzcd}$$

where c is the map that collapses $\frac{1}{2}S^k$. Show that the map $S^{k+1} \to Y$ defined by the maps β_0, H_{S^k} and β_1 extends to a map $D^{k+2} \to Y$.

(e) Why does this prove that Theorem 113(b) implies Theorem 113(c)?

Problem 9.21 Prove that Theorem 113(c) implies Theorem 113(d).

Now we show Theorem 113(d) implies Theorem 113(b).

Problem 9.22 For the purposes of this problem, assume Theorem 113(d), and let $k < n$.

(a) Prove that if $\alpha_1 : S^k \to (X, x_0)$ such that $f \circ \alpha_1 \simeq *$, then $\alpha_1 \simeq *$.

(b) Now let $\alpha_0 : S^{k+1} \to (Y, f(x_0))$. Show that there is $\alpha_1 : S^{k+1} \to (X, x_0)$ such that $f \circ \alpha_1 \simeq \alpha_0$.

Hint Consider the composition $D^{k+1} \xrightarrow{q} S^{k+1} \xrightarrow{\alpha_0} Y$, where q is the quotient map that collapses the boundary.

(c) Prove that Theorem 113(d) implies Theorem 113(b).

9.4.3 Equivalence of Parts (a), (c), and (e) of Theorem 113

PROBLEM 9.23 Show that Theorem 113(a) implies Theorem 113(c).

PROBLEM 9.24 Prove that Theorem 113(e) implies (Theorem 113(a) as follows.

(a) Assuming Theorem 113(e), let K be an n-dimensional CW complex, and prove that $f_* : [K, X] \to [K, Y]$ is surjective: let $\beta_0 : K \to Y$ be any (pointed) map, let $H : * \times I \to Y$ be the constant homotopy, and apply HELP to the diagram

$$\begin{tikzcd}
* \arrow[rr, hook] \arrow[dr] \arrow[dd, "\mathrm{in}_1"'] & & K \arrow[dl, dotted] \arrow[dd, "\mathrm{in}_1"] \\
& X & \\
(K \times 0) \cup (* \times I) \arrow[rr, hook] \arrow[dr, "\beta_0 \cup H"'] & & K \times I \arrow[dl, dotted] \\
& Y. &
\arrow[from=2-2, to=4-2, "f", crossing over]
\end{tikzcd}$$

(b) Assume Theorem 113(e) and prove that f_* is injective. Suppose $\theta, \phi : K \to X$ be homotopic with a pointed homotopy $Z : f \circ \theta \simeq f \circ \phi$; then apply HELP with the following data:

- $A = (K \times 0) \cup (* \times I) \cup (K \times 1)$,
- $B = K \times I$,
- $\alpha_1 = \theta \cup * \cup \phi$, and
- $H : (K \times S^0) \cup (* \times I) \to Y$ is the constant homotopy at $f \circ \alpha_1$.

The proof that Theorem 113(c) implies Theorem 113(e) will make use of our first induction principle for CW complexes. First we set up the Zorn's Lemma portion of the proof. Consider the set $\mathcal{P}$ of all triples (U, β_U, H_U) where U is a (relative) subcomplex of L such that $K \subseteq U \subseteq L$, and the maps β_U and H_U fit into the strictly commutative diagram

$$\begin{tikzcd}
K \arrow[rr, hook] \arrow[dr, "\alpha_1"] \arrow[dd, "\mathrm{in}_1"'] & & U \arrow[dl, "\beta_U"'] \arrow[dd, "\mathrm{in}_1"] \\
& X & \\
(U \times 0) \cup (K \times I) \arrow[rr, hook] \arrow[dr, "\beta_0|_U \cup H_K"'] & & U \times I \arrow[dl, "H_U"] \\
& Y &
\arrow[from=2-2, to=4-2, "f", crossing over]
\end{tikzcd}$$

is strictly commutative. Define a partial order on $\mathcal{P}$ by setting $(U, \beta_U, H_U) \leq (V, \beta_V, H_V)$ if $U \subseteq V$ and the diagram

$$\begin{tikzcd}
 & U \arrow[dl, "\beta_U"'] \arrow[dr] \arrow[dd, "\mathrm{in}_1"'] & \\
X \arrow[dd, "f"'] & & V \arrow[ll, "\beta_V"] \arrow[dd, "\mathrm{in}_1"] \\
 & U \times I \arrow[dl, "H_U"'] \arrow[dr] & \\
Y & & V \times I \arrow[ll, "H_V"']
\end{tikzcd}$$

is strictly commutative.

Problem 9.25

(a) Show that $\mathcal{P} \neq \varnothing$.

(b) Show that every chain $\cdots \leq (U, \beta_U, H_U) \leq (V, \beta_V, H_V) \leq \cdots$ of elements of $\mathcal{P}$, has an upper bound (Z, β_Z, H_Z).

(c) Use Zorn's Lemma to show there is a complex M maximal with the property that the theorem holds for the inclusion $j_M : K \hookrightarrow M$.

If $M = L$, we are done, so let us assume that M is a proper subcomplex. Then we can use Lemma 111 to find another subcomplex $N = M \cup_\lambda D^{k+1}$ (with $k < n$), which is the pushout in

$$\begin{tikzcd}
S^k \arrow[r, "\lambda"] \arrow[d] \arrow[dr, phantom, "\text{pushout}"] & M \arrow[d] \\
D^{k+1} \arrow[r, "\chi"] & N.
\end{tikzcd}$$

Problem 9.26 Use the diagram

$$\begin{tikzcd}
K \arrow[rr, hook] \arrow[dr, "\alpha_1"'] \arrow[dd, "\mathrm{in}_1"'] & & M \arrow[dl, "\beta_K"] \arrow[dd, "\mathrm{in}_1"] & S^k \arrow[l, "\lambda"'] \arrow[r] \arrow[dd] & D^{k+1} \arrow[dd] \\
 & X \arrow[dd, "f"] & & & \\
(N \times 0) \cup (K \times I) \arrow[rr, hook] \arrow[dr, "\beta_0|_N \cup H_K"'] & & (N \times 0) \cup (M \times I) \arrow[dl, "\beta_0|_N \cup H_M"] & (D^{k+1} \times 0) \cup (S^k \times I) \arrow[l] \arrow[r] & D^{k+1} \\
 & Y, & & &
\end{tikzcd}$$

to show that there are maps β_N and H_N so that $(N, \beta_N, H_N) > (M, \beta_M, H_M)$. Deduce that Theorem 113(c) implies Theorem 113(e).

9.5 The J.H.C. Whitehead Theorem

One of the most important consequences of Theorem 122 is that a weak equivalence $f : X \to Y$ induces isomorphisms $f_* : [K, X] \to [K, Y]$ for *all* CW complexes, not just the finite ones. This is known as the **J.H.C. Whitehead Theorem**.

Theorem 115 (J.H.C. Whitehead) *If $f : X \to Y$ is an ∞-equivalence then*

$$f_* : [K, X] \to [K, Y]$$

is an isomorphism for all CW complexes K.

PROBLEM 9.27 Prove the Whitehead theorem by adapting our proof that Theorem 113(e) implies Theorem 113(a).

This has an extremely important corollary: to determine whether a map of CW complexes is a homotopy equivalence, it suffices to check that it induces isomorphisms on homotopy groups.

Corollary 116 *If $f : X \to Y$ be a map of connected CW complexes, then the following are equivalent:*

1. *$f_* : \pi_*(X) \to \pi_*(Y)$ is an isomorphism*
2. *f is a homotopy equivalence.*

PROBLEM 9.28 Prove Corollary 116.

HINT This is a purely formal category-theoretical consequence of Theorem 115.

An ∞-connected space X is called **weakly contractible**.

EXERCISE 9.29

(a) Show that a weakly contractible CW complex is contractible.

(b) Are there any spaces that are weakly contractible but not contractible?

9.6 Things That Should Make You Wonder

1. We know that if the fiber of $f : X \to Y$ is $*$, then f is a homotopy equivalence. I the cofiber is $*$, then we know Σf is a homotopy equivalence. But what about f? Is it an equivalence?

2. What if the fiber of f is a n-connected? What does this say about the connectivity of the cofiber?

3. Suppose $A \to B \to C$ is a cofiber sequence and you know that two of the three spaces are n-connected. What does this imply about the third space?

These questions cannot be answered using the formal results we have developed so far, because (1) the notion of n-equivalence depends on special spaces, the spheres, and (2) they involve comparing domain type objects (cofibers) with target type objects (fibers), and this cannot be done formally.

Never fear! We will study questions such as these in great detail in the next part.

Chapter 10

Cellular Approximation of Maps

New Theme: *subdivision of cells.*

A CW complex is the colimit (and homotopy colimit) of the telescope diagram $X_0 \to X_1 \to \cdots \to X_n \to X_{n+1} \to \cdots$ of its skeleta. If X and Y are CW complexes and we have a strictly commutative diagram

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \\
Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & Y_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

then, on taking the colimit, we have an induced map $f : X \to Y$. Maps of this kind, which restrict to maps $f : X_n \to Y_n$ for each n, are called *cellular maps.* The amazing thing about CW complexes is that *every* map $X \to Y$ is homotopic to a cellular map. This fact has many surprisingly strong consequences, and it is one of the features that distinguishes homotopy theory in the topological category from abstract homotopy theory.

The proof is based on a simple fact about maps $f : I^k \to I^n$ with $k < n$. While it may happen that such a map is surjective, f will be homotopic, by a small homotopy, to a map which is not surjective.

To make use of the subdivision, we will constantly use The Lebesgue Number Lemma.

Theorem 117 (Lebesgue Number Lemma) *If X is a compact metric space with $X = U_1 \cup \cdots \cup U_n$ then there is $\delta > 0$ such that each open ball with diameter δ is completely contained in at least one U_i.*

Problem 10.1 Look up and work through the proof of the Lebesgue Number Lemma.

10.1 The Seifert-Van Kampen Theorem

We begin with an important theorem for the computation of fundamental groups. The proof uses the subdivision of squares into smaller squares, which is nice because it is easy to visualize. Given a homotopy pushout square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\
C \arrow[r] & D,
\end{tikzcd}$$

how does $\pi_1(D)$ relate to the fundamental groups of A, B and C? If you follow your and we apply π_1 to the diagram, you obtain

$$\begin{tikzcd}
\pi_1(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pushout of groups}"] & \pi_1(B) \arrow[d] \arrow[ddr, bend left] & \\
\pi_1(C) \arrow[r] \arrow[drr, bend right] & P \arrow[dr, "\rho"] & \\
 & & \pi_1(D),
\end{tikzcd}$$

where P is the pushout, in the category of group, of the prepushout diagram $\pi_1(C) \leftarrow \pi_1(A) \rightarrow \pi_1(B)$. The best thing we could possibly hope for is that the map ρ is an isomorphism. The Seifert-Van Kampen Theorem asserts that this is almost always true!

Theorem 118 (Seifert-Van Kampen Theorem) *If A is path-connected, then ρ is an isomorphism.*

The pushout of the prepushout diagram of groups $G \leftarrow A \rightarrow H$ is known in group theory as the **amalgamated product**, and denoted $G *_A H$. Thus the Seifert-Van Kampen Theorem is often written in the form

$$\pi_1(D) \cong \pi_1(B) *_{\pi_1(A)} \pi_1(C).$$

To build P, first form the **free product** $\pi_1(B) * \pi_1(C)$, which is the set of all 'words' $x_1 x_2 \cdots x_n$ where each x_i is in either G or in H; we are allowed to replace $x_1 x_2 \cdots x_i x_{i+1} \cdots x_n$ with $x_1 x_2 \cdots (x_i x_{i+1}) \cdots x_n$ if x_i and x_{i+1} are in the same group, so that their product is defined.

Exercise 10.2 Show that the free product of two groups is very poorly named: it is the sum in the category of groups.

Inside of the free product, we find the smallest normal subgroup $K \subseteq G * H$ containing all elements of the form $f_*(x)g_*(x^{-1})$ where $x \in A$. Then we define

$$P = (G * H)/K.$$

Note that each element of P has an expression as a product in which no two consecutive factors are from the same group; we'll call such an expression a standard form for the element.

Exercise 10.3 Check that P is indeed the pushout of the diagram $G \leftarrow A \rightarrow H$ in $\mathcal{G}$.

Exercise 10.4 Determine the pushout of $\{1\} \leftarrow H \rightarrow G$ in the category $\mathcal{G}$.

To prove Theorem 118, we first replace the given space D with the double mapping cylinder $C \leftarrow A \rightarrow B$: i.e., the space

$$\overline{D} = ((C \times 0) \cup (A \times I) \cup (B \times 1))/\sim$$

where $(a, 0) \sim (g(a), 0)$ and $(a, 1) \sim (f(a), 1)$. Let $U_0 \subseteq \overline{D}$ be the set of those points with $t \leq \frac{2}{3}$ and let $U_1 \subseteq \overline{D}$ be the set with $t \geq \frac{1}{3}$. Thus U_0 and U_1 are open, cover $\overline{D}$, and the inclusions

$$B \hookrightarrow U_1, \quad C \hookrightarrow U_0, \qquad \text{and} \qquad A \hookrightarrow U_0 \cap U_1$$

are homotopy equivalences.

Here's some notation that I think is helpful. Given a path $\alpha : I \rightarrow \overline{D}$, we want to construct a loop from it. Suppose we have previously decided on our favorite paths $\xi_0, \xi_1 : I \rightarrow U_0 \cap U_1$ from $*$ to $\alpha(0)$ and $\alpha(1)$, respectively. Then we can form the loop

$$\alpha^\circ = \xi_0 * \alpha * \overleftarrow{\xi_1}.$$

If we have not agreed on the paths ξ_0 and ξ_1, then the notation should not be used, since it suggests that α° depends only on α. Note also that it makes sense to use this notation for paths that are defined on intervals besides I; but we do assume that the loop α° is defined on I, which might require some conceptually easy but algebraically annoying reparametrization to write down explicitly.

Problem 10.5 Let $\alpha : I \rightarrow \overline{D}$ with $\alpha(0) = \alpha(1) = *$, so that α represents a class in $\pi_1(\overline{D})$. Divide I into n subintervals, $I_1, I_2, \ldots, I_n$ with endpoints $\frac{k}{n}$ for $k = 0, 1, \ldots, n$.

(a) Show that for large enough $n \in \mathbb{N}$, then α carries each subinterval entirely into U_0 or U_1 (or both).

(b) Identify those endpoints $\frac{k}{n}$ such that $\alpha(I_{k-1})$ and $\alpha(I_k)$ do not both lie in U or in V, let $0 = x_0 < x_1 < \cdots < x_m = 1$ be this set of points. Show that each $x_k \in U_0 \cap U_1$.

(c) Now we have a new subdivision of I into nonuniform intervals $J_k = [x_{k-1}, x_k]$. Let β_k be the restriction of α to J_k.

(d) Choose paths $\xi_k : I \to U_0 \cap U_1$ from $*$ to x_k. Then we can define $\beta_k^\circ = \xi_{k-1} * \beta_k, * \overleftarrow{\xi_k}$. Show that $[\alpha] = [\beta_1 * \beta_2 * \cdots * \beta_m] \in \pi_1(\overline{D})$.

(e) Conclude that $\rho : P \to \pi_1(\overline{D})$ is surjective.

This is already good enough to prove something useful.

Problem 10.6 Show that if X is path-connected then ΣX is simply-connected.

For more subtle computations, we need to know that ρ is also injective.

We want to show that our comparison map $\rho : P \to \pi_1(\overline{D})$ is injective. So let $[\alpha_1] \cdot [\alpha_2] \cdots [\alpha_n] \in P$ be a class, in standard form, that is mapped to $[*] \in \pi_1(D)$. Thus $[\alpha_1 * \alpha_2 * \cdots * \alpha_n] = [*] \in \pi_1(\overline{D})$. We have to show $[\alpha_1] \cdot [\alpha_2] \cdots [\alpha_n] = 1 \in P$.

By definition, $\rho([\alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n]) = [\alpha_1 * \alpha_2 * \cdots * \alpha_n]$. Let $H : I \times I \to \overline{D}$ be a nullhomotopy of this path. Subdivide the square $I \times I$ so finely that each subsquare is mapped entirely into U or V (or both). We may assume that each edge is broken into a number of nr pieces, where $r \geq 1$. Thus, each α_i is broken into some number of pieces, each of which occurs entirely in U_0 or entirely in U_1.

Now the square is cut into nk horizontal strips, each of height $\frac{1}{nk}$. In each strip, group together squares that are next to each other horizontally and which map into the same U_i. After the grouping, the strip will be decomposed into rectangles with height $\frac{1}{n}$ and various widths, each of which is mapped entirely into U_0 or entirely into U_1. Furthermore, adjacent rectangles are *not* mapped entirely into the same set. Now let's set up some notation.

1. Denote the x-coordinates of the vertical edges of the rectangles in the r^{th} strip by $0 = x_{0,r} < x_{1,r} < \cdots < x_{m,r} = 1$.

2. α_r can be decomposed in two ways:

   (a) thought of as the bottom of the r^{th} strip, it is $\alpha_{0,r} * \alpha_{1,r} * \cdots * \alpha_{m,r}$.

   (b) thought of as the top of the $(r-1)^{\text{st}}$ strip, it is $\beta_{0,r} * \beta_{1,r} * \cdots * \beta_{n,r}$.

3. also write $\nu_{i,r}$ for the restriction of H to the i^{th} vertical segment in the r^{th} strip (note that the first and the last of these are contant)

4. choose, once and for all, paths $\xi_{i,r}$ from $*$ to $\alpha_{i,r}(0)$; the first and the last we take to be constant paths. We may now form the loops $\alpha^{\circ}_{i,r} = \xi_{i,r} * \alpha_{i,r} * \overleftarrow{\alpha}_{i+1,r}$.

Problem 10.7

(a) Show by induction that, in the group P,

$$[\alpha^{\circ}_{0,r}] \cdot [\alpha^{\circ}_{1,r}] \cdots [\alpha^{\circ}_{m,r}] = [\beta^{\circ}_{0,r}] \cdot [\beta^{\circ}_{1,r}] \cdots [\beta^{\circ}_{i-1,r}] \cdot [\overleftarrow{\nu}^{\circ}_{i,r}] \cdot [\alpha^{\circ}_{i,r}] \cdots [\beta^{\circ}_{m,r}].$$

Conclude that in the group P, $[\alpha^{\circ}_{0,r}]\cdot[\alpha^{\circ}_{1,r}] \cdots [\alpha^{\circ}_{m,r}] = [\beta^{\circ}_{0,r}]\cdot[\beta^{\circ}_{1,r}] \cdots [\beta^{\circ}_{m,r}]$.

(b) Show $[\beta^{\circ}_{0,r}] \cdot [\beta^{\circ}_{1,r}] \cdots [\beta^{\circ}_{m,r}] = [\alpha^{\circ}_{0,r+1}] \cdot [\alpha^{\circ}_{1,r+1}] \cdots [\alpha^{\circ}_{n,r+1}] \in P$.

(c) Prove Theorem 118.

Exercise 10.8 Describe the sense in which the isomorphism of Theorem 118 is natural.

Problem 10.9 Generalize Theorem 118 to the situation where $X = \bigcup_{i \in \mathcal{I}} U_i$, where, for any $i \neq j \in \mathcal{I}$, $U_i \cap U_j = V$, a fixed path-connected open subset of X.

This theorem makes computation possible.

Problem 10.10 Show that $\pi_1(A \vee B) = \pi_1(A) * \pi_1(B)$. Generalize to arbitrary wedges.

Problem 10.11 Determine $\pi_1(M(\mathbb{Z}/a, 1))$.

Recall that a **presentation** of a group is an expression

$$G = \langle X \mid R \rangle$$

where X is a set and $R \subseteq F(X)$, the free group on X. The notation means that there is a map $X \to G$, such that the extension $F(X) \to G$ is onto, and whose kernel is $\overline{R}$, the smallest normal subgroup of $F(X)$ containing the set R. Thus the elements of X are called **generators** for G and the elements of R are called **relations** among those generators.

Exercise 10.12

(a) Write down $G *_A H$ in terms of generators and relations.

(b) Write down $\langle X \mid R \rangle$ as a pushout of groups.

We finish by showing that every group G arises as the fundamental group of a two-dimensional CW complex.

Problem 10.13 Let $G = \langle X \mid R \rangle$.

(a) Let $W = \bigvee_{x\in X} S^1$, and show that $\pi_1(W) \cong F(X)$.

(b) Let $V = \bigvee_{r\in R} S^1$, and find a map $\rho : V \to W$ such that $\mathrm{Im}(r_*)$ is the subgroup of $F(X)$ generated by the elements of R.

(c) Then consider the homotopy pushout square

$$\begin{tikzcd} V \arrow[r, "\rho"] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & W \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

and show that $\pi_1(X) \cong G$.

10.2 Simplices, Subdivision and Proof of Lemma 120

To apply this kind of argument in higher dimensions, we need to be able to decompose disks into smaller disks. We will introduce some new models for disks (and spheres), called *simplices* (and their boundaries). They have three crucial properties: they are homeomorphic to disks; they can be easily subdivided in to smaller simplices; and linear maps from simplices are defined and determined by their values on the vertices of the simplex.

Simplices and Their Boundaries. If $k \le m$, then a set $\{x_0, x_1, \dots, x_k\}$ of $k+1$ points in $\mathbb{R}^m$ is said to be in **general position** if $\{x_1 - x_0, x_2 - x_0, \dots, x_k - x_0\}$ is a linearly independent set of vectors. The k**-simplex** spanned by the points $x_0, x_1, \dots, x_k \in \mathbb{R}^m$, in general position (with $k \le m$) is the set

$$\Delta(x_0, x_1, \dots, x_k) = \left\{ \sum t_i x_i \;\middle|\; t_i \in I, \sum t_i = 1 \right\}.$$

The **boundary** of the simplex is

$$\partial\Delta(x_0, x_1, \dots, x_k) = \left\{ \sum t_i x_i \;\middle|\; t_i \in I, \sum t_i = 1, \text{ and at least one } t_i = 0 \right\}.$$

The standard k-simplex is $\Delta^k = \Delta(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_{k+1}) \subseteq \mathbb{R}^{k+1}$, where the $\mathbf{e}_i$ are the standard unit vectors. A map $\ell : \Delta(x_0, x_1, \dots, x_k) \to \mathbb{R}^n$ is called **linear** if $\ell(\sum t_i x_i\ ) = \sum t_i \ell(x_i)$.

EXERCISE 10.14 Draw the standard simplices Δ^0, Δ^1, Δ^2 and their boundaries.

The simplex $\Delta(x_0, x_1, \dots, x_k)$ contains many subsimplices: every subset $S \subseteq \{x_0, x_1, \dots, x_k\}$ gives rise to a simplex $\Delta(S) \subseteq \Delta(x_0, x_1, \dots, x_k)$.

Exercise 10.15 How many j-dimensional subsimplices does a k-dimensional simplex have?

We will use simplices and their boundaries as models for disks and spheres.

Problem 10.16

(a) Prove that there is a commutative square

$$\begin{tikzcd} \partial\Delta(x_0, x_1, \ldots, x_k) \arrow[r, "\cong"] \arrow[d] & S^{k-1} \arrow[d] \\ \Delta(x_0, x_1, \ldots, x_k) \arrow[r, "\cong"] & D^k. \end{tikzcd}$$

(b) Show that any map $\{x_0, x_1, \ldots, x_k\} \to \mathbb{R}^n$ extends to a unique linear map $\Delta(x_0, x_1, \ldots, x_k) \to \mathbb{R}^n$.

The first part of Problem 10.16 shows that the simplex and its boundary are suitable models for disks and spheres. The second part gives the basic property that makes them so useful: it is easy to specify maps from simplices, at least maps into vector spaces.

Subdivision of Simplices. Next we have to discuss subdivision of simplices. The **barycenter** of the simplex $\sigma = \Delta(x_0, x_1, \ldots, x_k)$ is

$$b_\sigma = \frac{1}{k+1} \sum x_i.$$

Note that if $\sigma = \Delta(x)$, then $b_\sigma = \sigma$. A **barycentric subsimplex** of $\Delta(x_0, x_1, \ldots, x_k)$ is $\Delta(b_{\tau_0}, b_{\tau_1}, \ldots, b\tau_j)$ where $\tau_0 \subseteq \tau_1 \subseteq \cdots \subseteq \tau_j$.

Exercise 10.17 Draw all the barycentric subsimplices of Δ^2 and Δ^3.

Proposition 119 *Let σ be any k-simplex. Then*

(a) σ is the union of its k-dimensional barycentric subsimplices,

(b) the intersection of two barycentric subsimplices of σ is also a barycentric subsimplex of σ, and

(c) the diameter of each barycentric subsimplex of σ is at most $\frac{k}{k+1}$ times the diameter of σ.

You should feel free to take this for granted. But the proof is not difficult, so you should also feel free to prove it.

Exercise 10.18 Prove Proposition 119.

The decompositon of σ as the union of its barycentric subsimplices is called the **barcyentric subdivision** of σ; it is denoted $\mathrm{sd}(\sigma)$. If we apply the process to each of the simplices in a subdivision of σ, we obtain the second subdivision $\mathrm{sd}^2(\sigma)$, and so on.

EXERCISE 10.19 Draw the second barycentric subdivision $\mathrm{sd}^2(\Delta^2)$.

PROBLEM 10.20 Let σ be a simplex and let $\epsilon > 0$. Then there is some n so that each simplex in the n^{th} barycentric subdivision $\mathrm{sd}^n(\sigma)$ has diameter at most ϵ.

10.3 The Connectivity of $X_n \hookrightarrow X$

How much of an equivalence is the inclusion $X \hookrightarrow Y$, where Y is obtained from X by attaching cells of dimension $n+1$ or higher? On the face of it, this is an intractible problem, since cell attachment is a domain-type construction, and connectivity of maps is a target-type concept. But, using the subdivision of disks, we can resolve this problem. Because of CW induction, it ultimately boils down to the following lemma.

Lemma 120 *For any space X, the inclusion $X \hookrightarrow X \cup D^{n+1}$ is an n-equivalence.*

In light of our work in the previous chapter, we see that it is enough to show that in the diagram

$$\begin{tikzcd}
\partial\Delta^k \arrow[r, "\alpha_1"] \arrow[d] & X \arrow[d] \\
\Delta^k \arrow[r, "\beta_0"'] \arrow[ur, dotted] & X \cup D^n
\end{tikzcd}$$

with $k < n$, the dotted arrow can be filled in so that the top triangle commutes on the nose, and the bottom triangle commutes up to a homotopy that is constant on $\partial\Delta^k$.

PROBLEM 10.21 Show that the Lemma will follow from the following statement:

> *Every $\beta_0 : \Delta^k \to X \cup D^n$ is homotopic, by a homotopy constant on $\partial\Delta^k$, to a map β such that* $\mathrm{int}(D^n) \not\subseteq \mathrm{Im}(\beta)$.

PROBLEM 10.22 In the situation of Lemma 120, let $U = X \cup (D^n - \frac{1}{3}D^n)$ and $V = \frac{2}{3}D^n$.

(a) Show that there are functions $u, v : D^k \to I$ such that

$$U = u^{-1}(0,1], \quad V = v^{-1}(0,1], \quad \tfrac{1}{3}D^n \subseteq v^{-1}(1),$$

and $u + v : X \to [0,2]$ is the constant function at 1.

(b) Show that there is a subdivision of Δ^k in which each simplex is mapped either entirely into U or entirely into V.

Let L_U be the union of those k-simplices in the decomposition of Problem 10.22(b) which map into U, and let L_V be the union of the k-simplices which map into V; write β_U and β_V for the restrictions of β to L_U and L_V. To summarize, we have the diagram of prepushouts

$$\begin{tikzcd}
L_U \arrow[d, "\beta_U"] & L_U \cap L_V \arrow[l] \arrow[r] \arrow[d, "\beta_{U\cap V}"] & L_V \arrow[d, "\beta_V"] \\
U & U \cap V \arrow[l] \arrow[r] & V
\end{tikzcd}$$

which induces the map $\beta : \Delta^k \to X \cup D^n$ on pushouts.

Identify the interior of D^n with $\mathbb{R}^n$, so that we may meaningfully talk about linear combinations of points in $\operatorname{int}(D^n)$. Let $\ell : L_V \to X \cup D^n$ be the piecewise linear map determined by the values of f_V on the vertices of L_V. Then define $\phi : \Delta^k \to X \cup D^n$ by the formula

$$\phi(x) = u(f(x))f(x) + v(f(x))\ell(x).$$

Exercise 10.23 This formula asks you to form linear combinations of points that are not in the disk D^n. Make sense of this.

Problem 10.24

(a) Show that ϕ is homotopic to f, by a homotopy which is constant on $\partial\Delta^k$.

(b) Show that, $\ell(L_V) \cap \frac{1}{3}D^n$ is contained in a union of (translates of) k-dimensional subspaces.

(c) Conclude that $D^n \not\subseteq \phi(\Delta^k)$ and prove Lemma 120.

This lemma implies a more generally useful statement.

Proposition 121 *If X is a CW complex, then the inclusion $X_n \hookrightarrow X$ is an n-equivalence.*

Problem 10.25 Use Lemma 120 to prove Proposition 121.

Hint Let $\mathcal{P}$ be the set of all subcomplexes U such that $X_n \subseteq U \subseteq X$ and $X_n \hookrightarrow U$ is an n-equivalence. Show that $X \in \mathcal{P}$.

10.4 Cellular Approximation of Maps

A map $f : X \to Y$ between two CW complexes is a **cellular map** if for each n, $f(X_n) \subseteq Y_n$. Our goal is to prove the *Cellular Approximation theorem.*

Theorem 122 (Cellular Approximation) *Let $f : X \to Y$ be a map of CW complexes, and assume that $f|_A$ is cellular, where $A \subseteq X$ is a subcomplex. Then there is a map $g : X \to Y$ such that*

(a) $f \simeq g$,

(b) g *is cellular , and*

(c) $g|_A = f|_A$.

If f is a pointed map, then we can arrange that g is also pointed and that the homotopy $H : f \simeq g$ is a pointed homotopy.

This is proved by CW induction and Zorn's Lemma. Let $\mathcal{P}$ be the set of triples (U, f_U, H_U) where

- $X_n \subseteq U \subseteq X$ is a subcomplex,
- $f_U : U \to Y$ is cellular with $f_U|_{X_n} = f|_{X_n}$, and
- $H_U : f_U \simeq f$.

Define a partial order on $\mathcal{P}$ by setting $(U, f_U, H_U) \leq (V, f_V, H_V)$ if

- $U \subseteq V$,
- $f_V|_U = f_U$, and
- $H_V|_{U\times I} = H_U|_{U\times I}$.

Certainly Theorem 122 will follow if you can show that there is a map ϕ and a homotopy $H : f \simeq \phi$ such that $(X, \phi, H) \in \mathcal{P}$.

PROBLEM 10.26

(a) Show that $\mathcal{P}$ has a maximal element (M, f_M, H_M), and

(b) Show that if $M \neq X$, then (M, f_M, H_M) is not maximal.

HINT Find a map $\phi_M : X \to Y$ such that $\phi_M \simeq f$ and $\phi_M|_M = f_M$. Find a complex $M \cup D^k \subseteq X$ and let $\beta = \phi_M|_{M_k \cup D^k}$. Use Lemma 120.

10.5 Homotopy Colimits and n-Equivalences

Finally, we use the subdivision technique to prove a a useful result generalizing the fact that a pointwise homotopy equivalence of prepushout diagrams induces a homotopy equivalence of their homotopy pushouts.

10.5.1 Homotopy Pushouts and Pullbacks

First, let's look at homotopy pullbacks, which are easier to analyze.

Problem 10.27 Show that a pointwise n-equivalence of prepullback diagrams induces an $(n-1)$-equivalence of homotopy pullbacks.

Now to the main result.

Proposition 123 *Let $\phi : F \to G$ be the map*

$$\begin{tikzcd} Z \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\ C & A \arrow[l] \arrow[r] & B \end{tikzcd}$$

of prepushout diagrams. Assume that each of the vertical maps is an n-equivalence, with $n \leq \infty$.[1] *Then the induced map f of homotopy pushouts is also an n-equivalence.*

Exercise 10.28 Show that it suffices to verify Proposition 123 for your favorite induced map.

Since we can choose any homotopy colimit construction we like, let's use the standard double mapping cylinder. Write $f : W \to D$ for the induced map of homotopy pushouts. According to Theorem 113, we have to show that in any diagram of the form

$$\begin{tikzcd} \partial\Delta^k \arrow[r, "\beta"] \arrow[d] & W \arrow[d, "f"] \\ \Delta^k \arrow[r, "A"'] \arrow[ur, dotted, "B"] & D \end{tikzcd}$$

with $k < n$, the dotted arrow can be filled in so that the upper triangle commutes on the nose, and the lower triangle commutes up to a homotopy which is constant on $\partial\Delta^k$.

Problem 10.29

(a) Using the standard homotopy colimit (i.e., the double mapping cylinder), show that W and D can be decomposed into pairs of open subsets

$$W = W_1 \cup W_2 \quad \text{and} \quad D = D_1 \cup D_2$$

so that $f(W_1) \subseteq D_1$, $f(W_2) \subseteq D_2$ and the three restricted maps

$$f_1 : W_1 \to D_1 \quad \text{and} \quad f_2 : W_2 \to W_2 \quad \text{and} \quad f_{12} : W_1 \cap W_2 \to D_1 \cap D_2$$

---

[1] so ϕ could be described as a pointwise n-equivalence

are n-equivalences.

(b) Show that Δ^k can be subdivided into simplices so small that each of them is mapped either entirely into D_1 or entirely into D_2.

(c) Prove Proposition 123.

Proposition 123 implies that the pushout of an n-equivalence is also an n-equivalence.

Problem 10.30 Consider the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"'] \\
C \arrow[r] & D
\end{tikzcd}$$

(a) Show that if the square is a homotopy pullback square and g is an n-equivalence, then f is also an n-equivalence.

(b) Show that if the square is a homotopy pushout square and f is an n-equivalence, then g is also an n-equivalence.

(c) Suppose the square is a homotopy pushout square and the spaces A, B and C are n-connected. Show that D is also n-connected.

(d) Suppose that the square is a homotopy pullback square and the spaces B, C and D are n-connected; show that A is $(n-1)$-connected.

10.5.2 Telescope Diagrams

Now we tackle the analogous question for telescope diagrams.

Proposition 124 *Consider the map of telescope diagrams*

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d, "f_0"] & X_1 \arrow[r] \arrow[d, "f_1"] & \cdots \arrow[r] & X_m \arrow[r] \arrow[d, "f_m"] & X_{m+1} \arrow[r] \arrow[d, "f_{m+1}"] & \cdots \\
Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_m \arrow[r] & Y_{m+1} \arrow[r] & \cdots,
\end{tikzcd}$$

with induced map $f : X \to Y$ of homotopy colimits. If each map f_m is an n-equivalence, then f is also an n-equivalence.

For the proof we use the standard construction outlined in Section **??**, in which each map $X_m \to X_{m+1}$ is replaced with its mapping cylinder (and similarly for $Y_m \to Y_{m+1}$). We index the mapping cylinder of $X_m \to X_{m+1}$ using the interval $[m, m+1]$, so that the colimits X and Y come equipped with continuous functions $\tau_X : X \to [0, \infty)$ and $\tau_Y : Y \to [0, \infty)$ making the diagram

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "\tau_X"'] & & Y \arrow[dl, "\tau_Y"] \\
& {[0,\infty)} &
\end{tikzcd}$$

commute. To prove Proposition 124, we consider the diagram

$$\begin{tikzcd}
\partial\Delta^k \arrow[r, "\beta"] \arrow[d] & X \arrow[d, "f"] \\
\Delta^k \arrow[r, "A"'] \arrow[ur, dotted, "B"] & Y
\end{tikzcd}$$

with $k < n$, and show that the dotted arrow can be filled in, subject to the usual restrictions.

Problem 10.31 Using the functions τ_X and τ_Y, show that there is a m large enough that the dashed maps diagram in the diagram

$$\begin{tikzcd}
\partial\Delta^k \arrow[rr, "\beta"] \arrow[dr, dashed] \arrow[dd] & & X \arrow[dd, "f"] \\
& X_m \arrow[ur] \arrow[dd, "f_m"] & \\
\Delta^k \arrow[rr, "A"] \arrow[ur, dotted] \arrow[dr, dashed] & & Y \\
& Y_m \arrow[ur] &
\end{tikzcd}$$

can be filled in so that the diagram commutes. Then prove Proposition 124.

Arguments such as this, where a question about a telescope is reduced to the same question about some finite stage of the diagram, are referred to as **small object arguments**. Abstractly, an object X in a category $\mathcal{C}$ is a **small object** with respect to a telescope diagram if each map from X into the colimit of a telescope diagram factors through a finite stage of the diagram. In the category of topological spaces, smallness usually is proved using compactness.

Problem 10.32

(a) Show that the condition on the maps f_m can be relaxed to f_m is an n-equivalence for sufficiently large m.

(b) Show that if the connectivity of f_m goes to ∞ as $m \to \infty$, then f is a weak equivalence.

(c) Show that if the connectivity of X_m goes to ∞ as $m \to \infty$, then X is weakly contractible.

Chapter 11

Patching Fibrations Together

The underlying theorem of this chapter is that a map which is locally a fibration (or a weak fibration) is actually a fibration (or a weak fibration). This theorem was first proved by Hurewicz, and later given its definitive form by Dold.

This theorem is recast as a rule for manipulating diagrams, called The First Cube Theorem. To state the theorem, we make a definition: a homotopy commutative cube is a **Mather cube** if the bottom and top faces are homotopy pushout squares and the sides are homotopy pullback squares. The First Cube Theorem states that if the top and bottom are homotopy pushout squares and the back and left are homotopy pullback squares, then the cube is a Mather cube.

To prove this, we need weak fibrations instead of fibrations. Like ordinary fibrations, they are sufficient to define homotopy pullbacks. But weak fibrations have an advantage over ordinary fibrations: a map homotopy equivalent to a weak fibration is also a weak fibration.

We begin with a brief study of weak fibrations. Then we state the theorem of Hurewicz and Dold, and finally we derive the First Cube Theorem.

11.1 Weak Fibrations

Ordinary fibrations are too rigid for our proof of the first cube theorem, so we introduce a more flexible notion of fibration that is better suited to our purposes. A **weak fibration** is a map $q : X \to Y$ that is homotopy equivalent, in the category $\mathrm{map}(\mathcal{T})$ of maps, to a fibration $p : E \to B$.

EXERCISE 11.1

(a) Show that every fibration is a weak fibration.

(b) Criticize the following argument:

> We can take any map $f : X \to Y$ and convert it to a fibration, yielding the strictly commutative square
>
> $$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d, "f"'] & E_f \arrow[d, "p"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$
>
> in which p is a fibration. Therefore, *every* map is homotopy equivalent to a fibration, and hence every map is a weak fibration.

(c) Let $Z = S^1 \cup * \times I \subseteq S^1 \times I$, and let $p : Z \to S^1$ be the restriction to Z of pr_1. Show that p is a weak fibration but not a fibration.

Whether or not f is a weak fibration depends only on the homotopy equivalence class of f in $\mathrm{map}(\mathcal{T})$. This is the property of weak fibrations that sets it apart from ordinary fibrations.

PROBLEM 11.2

(a) Show that if f is a weak fibration and g is homotopy equivalent to f in $\mathrm{map}(\mathcal{T})$, then g is also a weak fibration.

(b) Show that (a) is false if 'weak fibration' is replaced with 'fibration.'

PROBLEM 11.3 Let $f : X \to Y$ and convert it to a fibration, yielding the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & E_f \arrow[d, "p"] \\ Y \arrow[r, equal] & Y. \end{tikzcd}$$

Show that f is a weak fibration if and only if this square is a homotopy equivalence in $\mathrm{map}(\mathcal{T})$.

The class of weak fibrations is the smallest collection of maps that contains fibrations and is closed under homotopy equivalence in $\mathrm{map}(\mathcal{T})$. Being so closely related to fibrations, weak fibrations share many of the nice properties of fibrations. In particular, pulling back a weak fibration results in a homotopy pullback square.

Proposition 125 *If q is a weak fibration in the pullback square*

$$\begin{tikzcd} P \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\text{pullback}"] & X \arrow[d, "f"] \\ A \arrow[r] & B \end{tikzcd}$$

then

(a) *g is also a weak fibration, and*

(b) *the square is a strong homotopy pullback square.*

PROBLEM 11.4 Prove Proposition 125.

11.2 Pasting Fibrations Together

Suppose you have a function $p : E \to B$, and you wonder whether it is a fibration or not. If you have an open cover $B = \bigcup_\alpha U_\alpha$ together with lots of information about the individual subspaces U_α, then it makes sense to look at preimages $E_\alpha = p^{-1}(U_\alpha)$ and the restricted maps $p_\alpha : E_\alpha \to U_\alpha$. These maps can be constructed as pullbacks in the diagram

$$\begin{tikzcd} E_\alpha \arrow[r] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ U_\alpha \arrow[r] & B. \end{tikzcd}$$

If p were a fibration (or a weak fibration) then each p_α would also be a fibration (or a weak fibration).

But what about the reverse implication? If you only know that each individual map $p_\alpha : E_\alpha \to U_\alpha$ is a fibration, can you conclude that the original map p is a fibration? If all the pullback maps are weak fibrations and the cover $\{U_\alpha\}$ satisfies a very mild condition, then the reverse implication is perfectly valid, for both ordinary fibrations and for weak ones.

An open cover $\{U_\alpha\}$ of a space X is called **numerable** if there is a locally finite partition of unity subordinate to the cover. That is, if there are functions $\tau_\alpha : X \to [0, 1]$ such that

1. for each $x \in X$, $\tau_\alpha(x) = 0$ for all but finitely many α,
2. for each $x \in X$, the (finite!) sum $\sum_\alpha \tau_\alpha(x) = 1$, and
3. for each α, $\overline{\tau^{-1}((0, 1])} \subseteq U_\alpha$.

For many familiar spaces, every cover is numerable, or at least can be refined to a numerable one.

Theorem 126 *Suppose $p : E \to B$ is a map, and $B = \bigcup_\alpha U_\alpha$ is a numerable open cover. Then the following are equivalent:*

(a) each pullback map $p_\alpha : E_\alpha \to U_\alpha$ is a fibration (or weak fibration).

(b) $p : E \to B$ is a fibration (or weak fibration).

We already know that (b) implies (a), but the proof that (a) implies (b) is quite technical, and presenting the proof here would be a major interruption in the flow of ideas. Therefore, we suggest that you take it for granted for the purposes of this course.

Provenence of Theorem 126. Most people refer to a major paper of Dold [`?Dold1`] but this particular theorem, for ordinary fibrations, was proved by Hurewicz eight years earlier [`?Hurewicz1`]. Dold introduced weak fibrations, and refocused attention from properties of the space (paracompact) to properties of the cover (numerable), though this is present in Hurewicz's paper. Since Hurewicz, who defined fibrations, did not define weak fibrations, he did not prove the weak fibration part of Theorem 126, but his proof can be adapted for weak fibrations. A very thorough exposition of the proof (for ordinary fibrations) can be found in [`?Spanier`].

11.3 The First Cube Theorem

The first cube theorem is a homotopy-theoretic form of Theorem 126. In the category of topological spaces, the union of a collection of spaces is the prototypical example of a colimit, and the union of two spaces with specified intersection is the basic pushout. Our homotopy-theoretical version concerns the formation of homotopy pushouts (unions) of homotopy pullback squares (fibrations and pullbacks).

Theorem 127 (First Cube Theorem) *If, in the homotopy commutative cube*

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr] \arrow[dd] & & Z \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] & & D,
\end{tikzcd}$$

the top and bottom faces are homotopy pushout squares, $Z \to D$ is an induced map of homotopy pushouts, and the back and left faces are homotopy *pullback squares, then the cube is a Mather cube. That is, the front and right faces are also homotopy pullback squares.*

Problem 11.5 Show that it suffices to show that the conclusion about the front and right faces holds for a specific choice – your favorite one! – of induced map.

We will prove Theorem 127 by explicitly constructing a rather elaborate left nice approximation for the top of the cube. This replacement will have the two crucial properties: (1) the front and right faces of the replacement cube are categorical pullbacks, and (2) the induced map of homotopy pushouts is a weak fibration. Then Proposition 125(b) implies that the front and right faces are homotopy pullbacks, and the proof of the theorem will be complete.

The details are contained in the following problems.

Problem 11.6 Show that the homotopy commutative diagram

$$\begin{tikzcd}
Z \arrow[d] & X \arrow[l] \arrow[r] \arrow[d] & Y \arrow[d] \\
C & A \arrow[l] \arrow[r] & B
\end{tikzcd}$$

is pointwise homotopy equivalent in $\mathrm{h}\mathcal{T}$ to a diagram in which the vertical maps $Z \to C$ and $Y \to B$ are fibrations and the squares are strong homotopy pullbacks.

Since the new diagram is homotopy equivalent (in $\mathrm{h}\mathcal{T}$) to the given one, the cube we obtain by forming homotopy pushouts is pointwise homotopy equivalent in $\mathrm{h}\mathcal{T}$ to our other cube, and hence we may replace the original diagram with the new one you found, and not change the notation.

Now we can form the categorical pullbacks in the squares, and arrive at the strictly commutative diagram

$$\begin{tikzcd}
Z \arrow[d] & X_1 \arrow[l] \arrow[d] \arrow[dl, phantom, "\text{pullback}"] & X \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & X_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & Y \arrow[d] \\
C & A \arrow[l] \arrow[r, equal] & A \arrow[r, equal] & A \arrow[r] & B
\end{tikzcd}$$

in which the left and right squares are categorical pullbacks and the vertical maps, except for $X \to A$, are fibrations.

Problem 11.7 Show that there is strictly commutative diagram

$$\begin{tikzcd}
Z \arrow[d] & X_1 \arrow[l] \arrow[r, "\phi"] \arrow[d] \arrow[dl, phantom, "\text{pullback}"] & X_2 \arrow[r] \arrow[d] \arrow[dl, phantom, "\textcircled{\simeq}"] \arrow[dr, phantom, "\text{pullback}"] & Y \arrow[d] \\
C & A \arrow[l, "u"'] \arrow[r, equal] & A \arrow[r, "v"'] & B
\end{tikzcd}$$

in which ϕ is a homotopy equivalence in $\mathcal{T} \downarrow A$.

Next we set up our left nice replacements. First we construct some mapping cylinders parametrized on nonstandard intervals

$$\overline{C} = C \cup (A \times [0, \tfrac{2}{3}])/\sim \qquad \text{and} \qquad \overline{B} = B \cup (A \times [\tfrac{1}{3}, 1])/\sim,$$

where $A \times 0$ is identified with its image in C, and $A \times 1$ is identified with its image in B. Finally, let $\overline{A} = A \times [\frac{1}{3}, \frac{2}{3}]$ be their intersection.

Exercise 11.8 Verify that $\overline{C} \leftarrow \overline{A} \rightarrow \overline{B}$ is a left nice replacement for $C \leftarrow A \rightarrow B$.

The replacement for the top is more intricate. Form the categorical pullbacks

$$\begin{tikzcd} T(Z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & Z \arrow[d] \\ \overline{C} \arrow[r] & C \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} T(Y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & Y \arrow[d] \\ \overline{B} \arrow[r] & B \end{tikzcd}$$

Note that the maps $T(Z) \to \overline{C}$ and $T(Y) \to \overline{B}$ are fibrations. Let $M \to \overline{A}$ be the mapping cylinder of

$$\phi \times \text{id} : X_1 \times [\tfrac{1}{3}, \tfrac{2}{3}] \to X_2 \times [\tfrac{1}{3}, \tfrac{2}{3}]$$

over $\overline{A}$ (see Section A.2.3 for definitions and basic properties of mapping cylinders in categories of maps). Finally, define

$$\overline{Z} = T(Z) \cup M, \quad \overline{Y} = T(Y) \cup M \quad \text{and} \quad \overline{X} = M.$$

This is a pretty involved construction. Let's try to develop some intuition for it.

Exercise 11.9 Suppose the given diagram is

$$\begin{tikzcd} I \arrow[d] & I \arrow[l, "\text{id}"'] \arrow[r, "\text{id}"] \arrow[d] & I \arrow[d] \\ * & * \arrow[l] \arrow[r] & * \end{tikzcd}$$

(i.e., $X = Y = Z = [0,1]$ and $A = B = C = *$). Draw the resulting diagram

$$\begin{tikzcd} \overline{Z} \arrow[d] & \overline{X} \arrow[l] \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ \overline{C} & \overline{A} \arrow[l] \arrow[r] & \overline{B}. \end{tikzcd}$$

Show that the induced map of categorical pushouts is a weak fibration but not a fibration.

Now let's establish some basic properties of the construction.

Problem 11.10

(a) Show that there is a map of prepushout diagrams

$$\begin{tikzcd}\overline{Z} \arrow[d] & \overline{X} \arrow[l] \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ \overline{C} & \overline{A} \arrow[l] \arrow[r] & \overline{B}.\end{tikzcd}$$

in which both squares are categorical pullbacks.

(b) Show that the top and bottom rows of this diagram are left nice.

(c) Show that the induced map $\overline{W} \to \overline{D}$ of pushouts is a weak fibration.

Problem 11.11 Prove Theorem 127.

There is a little bit of extra information available in your proof.

Proposition 128 *A map of prepushout diagrams*

$$\begin{tikzcd}\overline{Z} \arrow[d] & \overline{X} \arrow[l] \arrow[r] \arrow[d] & \overline{Y} \arrow[d] \\ \overline{C} & \overline{A} \arrow[l] \arrow[r] & \overline{B}.\end{tikzcd}$$

in which both squares are categorical pullbacks and the maps vertical maps are weak fibrations can be completed to a Mather cube in which vertical faces are all strong homotopy pullback squraes.

Problem 11.12 Prove Proposition 128.

11.4 Important Examples of Fibrations

We use Theorem 126 to show that some maps that will be used constantly later are fibrations. To use it, we need to be able to show that open coverings are numerable. The best hammer for this job is paracompactness.

Problem 11.13 Show that if X is paracompact, then every open cover of X has a numerable refinement.

recall refinement

Covering Spaces. Recall that a **covering map** is a map $p : \widetilde{X} \to X$ with the property that each point $x \in X$ has an open neighborhood U such that $p^{-1}(U)$ is homeomorphic to a disjoint union $U \times D$ (where D is a discrete set),[1] and under this identification, p is just the projection on the first coordinate.

PROBLEM 11.14 Let X be a paracompact space.

(a) Show that every covering map $p : \widetilde{X} \to X$ is a fibration.

(b) Show that the exponential map $\exp : \mathbb{R} \to S^1$ given by $\exp(t) = e^{2\pi i t}$ is a covering map, and conclude that exp is a fibration. What is the fiber?

(c) Consider the space n^{th} real projective space $\mathbb{R}\mathrm{P}^n$, which is the quotient of S^n by the equivalence relation $x \sim -x$. Show that the quotient map $q : S^n \to \mathbb{R}\mathrm{P}^n$ is a covering map, and hence a fibration. What is the fiber?

PROBLEM 11.15 For which values of k is $\pi_k(\mathbb{R}\mathrm{P}^n) \cong \pi_k(S^n)$? Determine all the homotopy groups of S^1.

Fiber Bundles. A **trivial bundle** with fiber F is a projection map $\mathrm{pr} : X \times F \to X$, or any map $p : E \to B$ homeomorphic to it in the sense that there is a strictly commutative diagram

$$\begin{tikzcd}
X \times F \arrow[r, "\cong"] \arrow[d, "\mathrm{pr}"'] & E \arrow[d, "p"] \\
X \arrow[r, "\cong"] & B.
\end{tikzcd}$$

A **fiber bundle** is a map $f : E \to B$ which is *locally* a trivial bundle. This means that B has an open cover $B = \bigcup U_\alpha$ and in each pullback square

$$\begin{tikzcd}
E_\alpha \arrow[r] \arrow[d, "p_\alpha"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\
U_\alpha \arrow[r] & B
\end{tikzcd}$$

the map $E_\alpha \to U_\alpha$ is a trivial bundle.

EXERCISE 11.16

(a) Show that every covering map is a fiber bundle.

---

[1] Notice that $X \coprod X = X \times S^0$, and, more generally, if we give our indexing set $\mathcal{I}$ the discrete topology, then $\coprod_{i \in \mathcal{I}} X \cong X \times \mathcal{I}$.

(b) Show that if B is connected (not necessarily path connected), then any two fibers of the fiber bundle $p : E \to B$ are homeomorphic.

(c) Let M be the Möbius band. Show that there is a map $M \to S^1$ that is a nontrivial fiber bundle.

PROBLEM 11.17 Show that if $p : E \to B$ is a fiber bundle and B is paracompact, then p is a fibration.

Complex Projective Spaces. Let $S^1 \subseteq \mathbb{C}$ (note that S^1 is a group under multiplication). This group acts on $\mathbb{C}^{n+1}$ by coordinatewise multiplication, and if $z \in S^1$, $x \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$, then $z \cdot x \in S^{2n+1}$ too. Therefore we can define the n^{th} **complex projective space** by setting $\mathbb{C}\mathrm{P}^n = S^{2n+1}/S^1$. We write a typical element of $\mathbb{C}\mathrm{P}^n$ in the form $[x] = [x_1, x_2, \ldots, x_n, x_{n+1}]$, where $|x| = 1$ and $[x] = [y]$ if and only if there is a $z \in S^1$ such that $x = z \cdot y$.

PROBLEM 11.18 If $[x] \in \mathbb{C}\mathrm{P}^n$, then at least one entry is nonzero, so the sets

$$U_i = \{[x] \mid x_i \neq 0\} \qquad \text{for} \quad i = 1, 2, \ldots, n+1$$

form an open cover of $\mathbb{C}\mathrm{P}^n$.

(a) Show that if $[x] \in U_i$, then there is a unique $\dot{x} \in [x]$ such that $\dot{x}_i \in \mathbb{R}^+$.

(b) Show that if $[x] = [y]$ then there is a unique $z \in S^1$ such that $x = z \cdot y$.

(c) Let $p : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ be the quotient map and consider the pullback map p_i in the diagram

$$\begin{tikzcd}
E_i \arrow[r] \arrow[d, "p_i"'] \arrow[dr, phantom, "\text{pullback}"] & S^{2n+1} \arrow[d, "p"] \\
U_i \arrow[r] & \mathbb{C}\mathrm{P}^n.
\end{tikzcd}$$

Show that $E_i = \{x \in S^{2n+1} \mid x_i \neq 0\}$.

(d) Show that $\phi_i([x], z) = z \cdot \dot{x}$ defines a homeomorphism $\phi_i : U_i \times S^1 \to E_i$.

HINT First show it is a bijection.

(e) Show that $p : S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ is a fiber bundle, and hence a fibration, with fiber S^1.

You can identify the first projective spaces with spheres.

PROBLEM 11.19 Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. A point in $\mathbb{F}\mathrm{P}^1$ is an equivalence class $[x_1, x_2]$ with $x_1, x_2 \in \mathbb{F}$ not both zero. We can define a function $\phi : \mathbb{F}\mathrm{P}^1 \to \mathbb{F} \cup \{\infty\}$ by the rule $\phi([x_1, x_2]) = \frac{x_1}{x_2}$.

(a) Show that ϕ defines a homeomorphism $\mathbb{R}\mathrm{P}^1 \cong S^1$.

(b) Show that ϕ defines a homeomorphism $\mathbb{C}\mathrm{P}^1 \cong S^2$.

(c) Conclude that there is a fibration sequence $S^1 \to S^3 \to S^2$.

HINT Use Sterographic Projection.

There are quaternionic projective spaces too. Recall that the quaternions $\mathbb{H}$ is simply $\mathbb{R}^4$ with a tricky multiplicative structure. This structure is such that $S^3 \subseteq \mathbb{H}$ is a multiplicative group which acts on $S^{4n+3} \subseteq \mathbb{H}^{n+1}$. The quotient by this action is called the n^{th} **quaternionic projective space**, and it is denoted $\mathbb{H}\mathrm{P}^n$. The canonical quotient map $S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ can be shown to be a fibration in the same way as in the previous problem.

EXERCISE 11.20 Show that $\mathbb{H}\mathrm{P}^1 \cong S^4$, and conclude that there is a fibration sequence $S^3 \to S^7 \to S^4$.

We will come back to the projective spaces in much greater detail in Chapter **??**.

Group Actions. Projective spaces are obtained by forming orbit spaces of group actions. Suppose the group (topological) G acts on the space X. Then we may form the space of orbits of the action, and wonder about the canonical quotient map $q : X \to X/G$. Is this map a fibration?

EXERCISE 11.21 Let $G = S^1$ act on $X = D^2$ by rotation (or by multiplication in $\mathbb{C}$, if you prefer). Determine the map $D^2 \to D^2/S^1$, and show that it is not a fibration.

We clearly need to impose some conditions on our action in order for the quotient map to be well behaved. Let's say an action $\alpha : G \times X \to X$ is **properly effective**[2] if each point $x \in X$ has a neighborhood U such that the restriction

$$\alpha|_{G\times U} : G \times U \to X$$

is an embedding.

PROBLEM 11.22 Show that if $\alpha : G \times X \to X$ is a properly effective action, then the quotient map $X \to X/G$ is a fibration with fiber G.

Effective actions are sometimes too much to hope for. Let $x \in X$, and let H be the isotropy subgroup of x. We can loosen things up a bit by asking for a neighborhood U of x so that the action induces an embedding

$$(G/H) \times U \to X.$$

PROBLEM 11.23 Show that if the action satisfies this property, then $q : X \to X/G$ is a fibration.

---

[2] temporary terminology until I get a chance to look up what it's actually called.

Problem 11.24 Show that there is a fibration sequence

$$\mathbb{R}\mathrm{P}^1 \to \mathbb{R}\mathrm{P}^{2n+1} \to \mathbb{C}\mathrm{P}^n.$$

Chapter 12

Pullbacks of Cofibrations

The underlying theorem of this chapter is Theorem 129, which says that if you form a pullback using a fibration and a cofibration, the induced maps from the pullback are a fibration and – amazingly! – a cofibration. The Second Cube Theorem reinterprets Theorem 129 in terms of Mather cubes: if the bottom of a cube is a strong homotopy pushout square and the sides are homotopy pullback squares, then the cube is a Mather cube. (i.e., the top is a homotopy pushout square).

12.1 Pulling Back a Cofibration

Nothing can be said formally about the pullback of a cofibration, because cofibrations are domain-type maps and pullbacks are target-type constructions. But in the category of topological spaces, the pullback of a cofibration by a fibration is a cofibration!

Theorem 129 *Let $p : E \to B$ be a fibration, and let $i : A \hookrightarrow B$ be a closed cofibration. Then in the pullback square*

$$\begin{tikzcd} E_A \arrow[r, "j"] \arrow[d, "q"'] \arrow[dr, phantom, "\text{pullback}"] & E \arrow[d, "p"] \\ A \arrow[r, "i"] & B \end{tikzcd}$$

the map $j : E_A \to E$ is also a cofibration.

To prove our pullback result, we need yet another characterization of cofibations.

Lemma 130 *A map $i : A \to X$ is a cofibration if and only if there is a map $u : X \to I$ such that $u(A) = \{0\}$ and a homotopy $H : X \times I \to X$ such that the diagram*

$$\begin{tikzcd}
A \arrow[r, "\mathrm{in}_0"] \arrow[d, "i"'] & A \times I \arrow[d, "i \times \mathrm{id}"] \arrow[ddr, bend left, "\mathrm{STATIC}_i"] & \\
X \arrow[r, "\mathrm{in}_0"] \arrow[drr, bend right, "\mathrm{id}_X"'] & X \times I \arrow[dr, "H"] & \\
 & & X
\end{tikzcd}$$

commutes and $H(x,t) \in A$ whenever $t > u(x)$.

Problem 12.1 Recall that i is a cofibration if and only if there is a retraction $r : X \times I \to X \cup (A \times I)$.

(a) Supposing i is a cofibration, write $r(x,t) = (r_1(x,t), r_2(x,t))$, and define $u(x) = \sup\{t - r_2(x,t) \,|\, t \in I\}$. Find a homotopy $H : X \times I \to X$ so that, together, H and u satisfy conditions of Lemma 130.

(b) Now suppose that u and H satisfy those conditions. Show that the function

$$r(x,t) = \begin{cases} (H(x,t), 0) & \text{if } t \leq u(x) \\ (H(x,t), t - u(x)) & \text{if } t \geq u(x) \end{cases}$$

defines a retraction of $X \times I \to X \cup (A \times I)$.

Now we proceed to the proof of Theorem 129. Supposing $i : A \hookrightarrow B$ is a cofibration, we can find a map $u : B \to I$ and a homotopy $H : B \times I \to B$ satisfying the conditions of Lemma 130. Consider the diagram

$$\begin{tikzcd}
E \arrow[rr, "\mathrm{id}"] \arrow[d, "\mathrm{in}_0"'] & & E \arrow[d, "p"] \\
E \times I \arrow[r, "p \times \mathrm{id}"] \arrow[urr, dotted, "\widehat{H}"] & B \times I \arrow[r, "H"] & B.
\end{tikzcd}$$

Since p is a fibration, the lift $\widehat{H}$ exists. Let $v = u \circ p : E \to I$ and define $K : E \times I \to E$ by the rule

$$K(x,t) = \begin{cases} \widehat{H}(x,t) & \text{if } t \leq v(x) \\ \widehat{H}(x, v(x))) & \text{if } t \geq v(x). \end{cases}$$

Problem 12.2 Prove Theorem 129 by showing the functions K and v satisfy the conditions of Lemma 130.

12.2 The Second Cube Theorem

The second cube theorem turns Theorem 129 into a powerful tool for manipulating homotopy pullbacks and pushouts.

As a warm-up exercise, let's analyze the pullbacks of a fibration over a very strong kind of homotopy pushout square. Suppose $X = A \cup B$, where A and B are closed in X (or open in X), so that X is the categorical pushout in the diagram

$$\begin{tikzcd}
A \cap B \arrow[r] \arrow[d] & A \arrow[d] \\
B \arrow[r] & X.
\end{tikzcd}$$

We further assume that each map in this square is a cofibration, so the square is a (strong) homotopy pushout square. Now let $p : E \to X$ be a fibration, and form the pullbacks

$$\begin{tikzcd}
E_{A\cap B} \arrow[r] \arrow[d, "p_{A\cap B}"'] \arrow[dr, phantom, "PB"] & E \arrow[d, "p"] & & E_A \arrow[r] \arrow[d, "p_A"'] \arrow[dr, phantom, "PB"] & E \arrow[d, "p"] & & E_B \arrow[r] \arrow[d, "p_B"'] \arrow[dr, phantom, "PB"] & E \arrow[d, "p"] \\
A \cap B \arrow[r] & X & & A \arrow[r] & X & & B \arrow[r] & X.
\end{tikzcd}$$

We can assemble these squares into a cube:

$$\begin{tikzcd}
E_{A\cap B} \arrow[rr] \arrow[dr] \arrow[dd] & & E_A \arrow[dr] \arrow[dd] & \\
& E_B \arrow[rr, crossing over] & & E \arrow[dd] \\
A \cap B \arrow[rr] \arrow[dr] & & A \arrow[dr] & \\
& B \arrow[rr] \arrow[from=uu, crossing over] & & X.
\end{tikzcd}$$

Problem 12.3 Show that the square

$$\begin{tikzcd}
E_{A\cap B} \arrow[r] \arrow[d] & E_A \arrow[d] \\
E_B \arrow[r] & E
\end{tikzcd}$$

is also a strong homotopy pushout square.

The Second Cube Theorem generalizes this simple argument to homotopy commutative cubes in which the base is a strong homotopy pushout square and the sides are ordinary homotopy pullback squares. Here is the statement.

Theorem 131 (Second Cube Theorem) *Suppose that in the homotopy commutative cube*

$$\begin{tikzcd}
W \arrow[rr] \arrow[dr] \arrow[dd] & & X \arrow[dr] \arrow[dd] & \\
& Y \arrow[rr, crossing over] & & Z \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] \arrow[from=uu, crossing over] & & D.
\end{tikzcd}$$

the bottom is a strong homotopy pushout square, and the sides are homotopy pullback squares. Then the top square is also a homotopy pushout square.

Problem 12.4

(a) Show that the given cube is pointwise homotopy equivalent in h$\mathcal{T}$ to a strictly commutative cube with the same base but in which all the vertical maps are fibrations, and all vertical squares are categorical pullbacks.

(b) Show that the given cube is pointwise homotopy equivalent in h$\mathcal{T}$ to a strictly commutative cube whose base is a categorical pushout in which all four maps are cofibrations, all the vertical maps are fibrations, and all vertical squares are categorical (hence homotopy) pullbacks.

(c) Prove Theorem 131.

Exercise 12.5 Can we get by with a simple homotopy pushout for the bottom?

Part III

Comparing Domains with Targets

Chapter 13

Approximations of Spaces

We feel pretty good about CW complexes, but sometimes we have to study spaces that are not CW complexes, or maybe not even homotopy equivalent to CW complexes. To apply our results to these other spaces, we need to be able to approximate an arbitrary space by a CW complex.

We define an n**-skeleton** of a space X to be an n-equivalence $K \to X$ in which K is an n-dimensional CW complex. An ∞-skeleton would then be a map $K \to X$ from a CW complex K which is a weak homotopy equivalence. The usual terminology is to call such a map a **CW replacement** for X. We show that every space X has a CW replacement, and hence has n-skeleta for each n.

Then we turn to the dual situation. The dual of n-dimensional is n-anticonnected. A space X is n-anticonnected if $[K, X] = *$ for all n-connected CW complexes K. A **Postnikov section** of a space X is an n-equivalence $X \to P$ where P is n-anticonnected.

13.1 Cellular Replacement of Spaces

An n**-skeleton** for a space X is an n-equivalence $X_n \to X$, where X_n is an n-dimensional CW complex. We prove some basic properties of skeleta and show that every space has n-skeleta for every n, including $n = \infty$.

An ∞-skeleton is simply a weak equivalence $\overline{X} \to X$ where $\overline{X}$ is a CW complex. This is more widely known as a **CW replacement** for X.

Exercise 13.1

(a) Explain in the language of an introductory topology course, what exactly is a 0-skeleton for a space X? Show that every space X has a 0-skeleton.

(b) Give an example to show that n-skeleta are not unique. Can you define a functor $Z : \mathcal{T}_* \to \mathcal{T}_*$ and a natural transformation $Z \to \mathrm{id}$ so that for every $X \in \mathcal{T}_*$, the map $Z(X) \to X$ is a 0-skeleton?

(c) Show that if X is already a CW complex then any CW replacement $\overline{X} \to X$ must be a homotopy equivalence.

PROBLEM 13.2 Suppose $i : X_n \to X$ is an n-skeleton of X and $j : X_m \to X$ is an m-skeleton for X, where $n \le m$. Show that there is a map $s : X_n \to X_m$ so that the diagram

$$\begin{tikzcd}
X_n \arrow[rr, "s"] \arrow[dr, "i"'] & & X_m \arrow[dl, "j"] \\
& X &
\end{tikzcd}$$

is strictly commutative. Show that s is an n-skeleton of X_m if $m > n$, but might only be an $(n-1)$-skeleton if $m = n$.

Now we want to show that every space has n-skeleta for every n.

EXERCISE 13.3 Show that if $\overline{X} \to X$ is a CW replacement for X, then the composite $\overline{X}_n \to \overline{X} \to X$ is an n-skeleton of X. Conclude that if X has an n-skeleton, then it has k-skeleta for all $k \le n$.

In view of this exercise, the following **Cellular Replacement Theorem** implies that every space X has n-skeleta for all $n \le \infty$.

Theorem 132 (Cellular Replacement) *Let $X_n \to X$ be an n-skeleton of the space X. Then there is a CW replacement $\overline{X} \to X$ such that $\overline{X}_n = X_n$.*

PROBLEM 13.4 Suppose X has an n-skeleton $i : X_n \to X$.

(a) Let $K = \ker(\pi_n(X_n) \to \pi_n(X))$ and define Y_{n+1} to be the $(n+1)$-dimensional CW complex defined by the pushout square[1]

$$\begin{tikzcd}
\coprod_{\alpha\in K} S^n \arrow[r] \arrow[d, "(\alpha)"'] \arrow[dr, phantom, "\text{pushout}"] & \coprod_{\alpha\in K} D^{n+1} \arrow[d] \\
X_n \arrow[r] & Y_{n+1}.
\end{tikzcd}$$

Show that there is a map $j_{n+1} : Y_{n+1} \to X$ whose induced map $\pi_k(Y_n) \to \pi_k(X)$ is an isomorphism for $k \le n$.

(b) Construct an $(n+1)$-skeleton $i_{n+1} : X_{n+1} \to X$.

HINT Use Problem **??**.

---

[1] In this diagram, (α) denotes the map which, in the α^{th} coordinate is a chosen map in the homotopy class of α.

In the construction of Problem 13.4, we have inclusions of subcomplexes $X_n \hookrightarrow X_{n+1}$ making the diagram

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & X_{n+r} \arrow[r] \arrow[d] & X_{n+r+1} \arrow[r] \arrow[d] & \cdots \\
X \arrow[r, equal] & X \arrow[r, equal] & \cdots \arrow[r, equal] & X \arrow[r, equal] & X \arrow[r, equal] & \cdots
\end{tikzcd}$$

strictly commutative. Let $\overline{X}$ be the colimit of the top row; then the diagram gives us an induced map $\overline{X} \to X$.

Problem 13.5

(a) Show that $\overline{X}$ is a CW complex.

(b) Prove the Cellular Replacement Theorem by showing that the induced map $\overline{X} \to X$ is a weak equivalence.

Next we study the functoriality and uniqueness of cellular replacements.

Theorem 133 *Let $f : X \to Y$, and let $i : \overline{X} \to X$ and $j : \overline{Y} \to Y$ be CW replacements for X and Y, respectively. Then*

(a) there is a map $g : \overline{X} \to \overline{Y}$ such that the diagram

$$\begin{tikzcd}
\overline{X} \arrow[r, "g"] \arrow[d, "i"'] & \overline{Y} \arrow[d, "j"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commutes up to homotopy,[2] *and*

(b) if g and g' are any two choices for such a map, then $g \simeq g'$; that is, the map g is uniquely determined up to homotopy.

Corollary 134 *If $\overline{X}$ and $\widetilde{X}$ are two CW replacements for the same space X, then $\overline{X} \simeq \widetilde{X}$.*

Problem 13.6 Prove Theorem 133 and Corollary 134.

Hint Use Corollary 116.

Functorial CW Replacement. Theorem 133 comes close to saying that CW replacement is functorial. But it does not quite say that, because the map g is only well defined up to homotopy, and because of the many choices made in the construction of $\overline{X}$ and $\overline{Y}$.

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[2] Can it be chosen to commute on the nose?

Exercise 13.7 Show that by making choices of CW replacements $\overline{X} \to X$ for each X, our construction can be used to define a functorial CW replacement whose domain is the homotopy category $\mathrm{H}\mathcal{T}$.

We end this section by outlining a functorial CW replacement in the category $\mathcal{T}$. This has the advantage of being functorial, but also the disadvantage of requiring infinitely many cells of all dimensions, even when the original space is already a CW complex!

Problem 13.8 Modify the proof of Theorem 132 by replacing the set K with the set $\widehat{K} = \{\alpha \in \mathrm{map}(S^n, \overline{X}_n) \mid i_n \circ \alpha \simeq *\}$.

(a) Check that the proof works equally well with this modified construction.

(b) Find a functorial way to extend a map $f : X \to Y$ to a strictly commutative square

$$\begin{tikzcd}
\overline{X} \arrow[r, "\bar{f}"] \arrow[d] & \overline{Y} \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

(that is, verify that $\bar{g} \circ \bar{f} = \overline{g \circ f}$).

Exercise 13.9 Show that in this modified construction, if $\overline{X}_n$ is not discrete, then $\widehat{K}$ is uncountable.

13.2 Connectivity and CW Structure

The connectivity of a space X is defined in terms of maps into X, which means that it is a target-type concept. Therefore it is comparatively easy to keep track of what happens to connectivity when target-type constructions are applied to a space. On the other hand, what happens to the connectivity when we apply domain-type constructions is far from obvious. We will characterize connectivity in terms of CW structure, which makes determining the effect of domain-type constructions on connectivity comparatively easy.

Connectivity and Cellular Structure. Theorem 132 implies that $(n-1)$-connected spaces are weakly equivalent to CW complexes with particularly simple $(n-1)$ skeletons. This fact allows us to estimate the connectivity of spaces formed by domain-type constructions.

Proposition 135 *Let $f : X \to Y$, and let $\overline{X} \to X$ be a CW replacement for X. Show that the following are equivalent:*

1. *f is an n-equivalence*

2. *Y has a CW replacement $\overline{Y} \to Y$ such that $\overline{X} \subseteq \overline{Y}$ is a subcomplex with $\overline{Y}_n = \overline{X}_n$ and the diagram*

$$\begin{tikzcd}
\overline{X}_n \arrow[r] \arrow[rd] & \overline{X} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
 & \overline{Y} \arrow[r] & Y
\end{tikzcd}$$

is strictly commutative.

Problem 13.10 Prove Proposition 135.

Corollary 136 *A space X is n-connected if and only if X has a CW replacement $\overline{X} \to X$ with $\overline{X}_n = *$.*

Problem 13.11 Prove Corollary 136.

Problem 13.12 Show that if X is a CW complex that is n-connected for every n, then $X \simeq *$.

In view of Problem 13.12, highly connected spaces should be thought of as being more nearly contractible. We often think of a map $X \to Y$ as a comparison between X and Y, and the homotopy fiber measures the difference between them. Thus the more highly connected the fiber is, the closer the map is to being an equivalence.

Connectivity and Domain-Type Constructions. Let's take a close look at the connectivity of spaces built from other spaces by domain-type constructions. These connectivity estimates will be used over and over in the following chapters.

Problem 13.13 Let X be an n-connected space and let Y be m-connected.

(a) What is the connectivity of $X \times Y$?

(b) What is the connectivity of $X \wedge Y$?

(c) What is the connectivity of ΣX?

(d) What is the connectivity of $X * Y$?

(e) What is the connectivity of $X \rtimes Y$?

Problem 13.14 Suppose f is an n-equivalence and g is an m-equivalence. Then how much of an equivalence is $f \wedge g$? What if $g = \mathrm{id}_B$ for some space B? What does this tell you about the connectivity of Σf?

Exercise 13.15 What is the connectivity of $\Omega^n \Sigma^m X$?

Here is a very important estimate.

PROBLEM 13.16 Suppose A is $(a-1)$-connected and B is $(b-1)$-connected. How highly connected is the inclusion $A \vee B \hookrightarrow A \times B$?

PROBLEM 13.17 Suppose A is $(a-1)$-connected. Then what is the connectivity of $\bigvee_{i\in\mathcal{I}} A \hookrightarrow \prod_{i\in\mathcal{I}} A$ (where the product is the weak product)?

Some Applications to H-Spaces and co-H-Spaces. Our first application is to the algebraic structure of $[A, X]$ when X is an H-space. We have shown that this set has a multiplication whose unit is the trivial map. We have not shown that this multiplication is associative, or that elements of $[A, X]$ have inverses.

PROBLEM 13.18 Let X be a an H-space.

(a) Show that, under the identification $\pi_*(X \times X) \cong \pi_*(X) \times \pi_*(X)$, the map $\mu_* : \pi_*(X) \times \pi_*(X) \to \pi_*(X)$ induced by the multiplication $\mu : X \times X \to X$ is given by $\mu_*(\alpha, \beta) = \alpha + \beta$.

(b) Show that the shear map $s : X \times X \to X \times X$ given by $s : (x_1, x_2) \mapsto (x_1, x_1x_2)$ is a weak homotopy equivalence.

(c) Show that for any CW complex A, if $\alpha, \beta \in [A, X]$, then there exists a unique $\xi \in [A, X]$ such that $\alpha\xi = \beta$.

(d) Also show that there is a unique $\zeta \in [A, X]$ such that $\zeta\alpha = \beta$.

Parts (c) and (d) of Problem 13.18 state that if A is a CW complex, then $[A, X]$ is an **algebraic loop**.

EXERCISE 13.19 Show that if L is an algebraic loop, then every element $\alpha \in L$ has a left inverse and a right inverse. Show that if L is associative, then the left and right inverses must be equal, and give an example to show that they need not be equal in general.

It is indeed the case that there are H-spaces for which $[A, X]$ is a nonassociative algebraic loop with distinct left and right inverses.

Next we see that a space whose dimension is small in comparison to its connectivity must be a co-H-space.

PROBLEM 13.20 Show that if X is an $(n-1)$-connected CW complex with $\dim(X) < 2n$, then X is a co-H-space.

Since we know that a co-H-space is a homotopy retract of a suspension, Problem 13.20(b) shows that there is a space Y such that X is a homotopy retract of ΣY. It does not assert any relation between the cells of Y and the cells of X – indeed, it does not even claim that Y is a CW complex! But it does suggest a question: are the attaching maps of the cells in X actually the suspensions of other maps? Is X itself a suspension?

13.3 Basic Obstruction Theory

Suppose you wish to construct a map $X \to Y$ satisfying certain properties, such as making a given diagram commute, or commute up to homotopy. If X is a CW complex, then you might try to construct the desired map by induction on the skeleta, or the cells, of X. The inductive step of such a construction reduces to the case $X = W \cup D^n$, where we have $f : W \to Y$, and we wish to extend f to a map $\phi : X \to Y$ making the diagram

$$\begin{tikzcd}
S^{n-1} \arrow[d, "i"'] \arrow[dr, bend left, "f \circ i"] & \\
W \arrow[r, "f"] \arrow[d, "j"'] & Y \\
X \arrow[ur, dotted, bend right, "\exists ? \phi"'] &
\end{tikzcd}$$

commute, at least up to homotopy. From the exactness of the sequence

$$\begin{tikzcd}
{[S^{n-1}, Y]} & {[W, Y]} \arrow[l, "i^*"'] & {[X, Y]}, \arrow[l, "j^*"']
\end{tikzcd}$$

we see that the map ϕ exists if and only if $f \circ i = 0 \in \pi_{n-1}(Y)$. The element $f \circ i \in \pi_{n-1}(Y)$ is called the **obstruction** to extending the map f, and so we say that the map f can be extended if and only if the obstruction is zero. This is why the technique is called **obstruction theory**.

EXERCISE 13.21 Formulate the dual obstruction theory problem.

Obstruction theory can be used very effectively to study the maps from a CW complex into a space whose homotopy groups vanish in large dimensions.

Proposition 137 *Let Y be a space with $\pi_k(Y) = 0$ for all $k \geq n$, and let X be a CW complex. Suppose $Z \subseteq X$ is a subcomplex with $X_n \subseteq Z \subseteq X$. Then for any map $f : Z \to Y$, the dotted arrow in*

$$\begin{tikzcd}
Z \arrow[rr] \arrow[dr, "f"'] & & X \arrow[dl, dotted, "\phi"] \\
& Y &
\end{tikzcd}$$

can be filled in so that the triangle is strictly commutative.

PROBLEM 13.22

(a) Let $\mathcal{P}$ denote the set of pairs (U, f_U), where U is a subcomplex of X with $Z \subseteq U \subseteq X$ and $f_U : U \to Y$ extends f. Define a partial order on the set $\mathcal{P}$, and show that $\mathcal{P}$ contains a maximal element.

(b) Show that if $U \neq X$, then (U, f_U) is not maximal, and thereby prove Proposition 137.

Proposition 137 can obviously be used to show that certain maps exist. But since a homotopy is also a map, the proposition can also be used to show that two given maps are homotopic to one another. This application relies on Lemma 26, which describes the cellular structure of the product of a CW complex with the interval I.

Proposition 138 *If Y is a space with $\pi_k(Y) = 0$ for all $k > n$, then for any CW complex X,*

(a) the inclusion $X_n \to X$ induces an injection $[X_n, Y] \leftarrow [X, Y]$, and

(b) If $m > n$, then the inclusion $X_m \to X$ induces an isomorphism $[X_m, Y] \leftarrow [X, Y]$.

Problem 13.23 Prove Proposition 138.

Hint For (a), use Problem **??**.

We have now proved enough to derive the dual of Problem 13.20(b).

Problem 13.24 Let X be a CW complex. Show that if X is $(n-1)$-connected and $\pi_k(X) = 0$ for $k \geq 2n - 1$, then X is an H-space.

13.4 Postnikov Sections

In Section 13.1 we repeatedly attached cells to construct a CW complex weakly equivalent to a given space. Now we use the same basic construction, but this time, we use it to mutilate the given space, rendering trivial all of its homotopy groups above a specified dimension.

Let X be any space, and let $n \geq 1$. A map $p : X \to Q$ is called an n^{th} **Postnikov approximation** if

1. $p_* : \pi_k(X) \to \pi_k(Q)$ is an isomorphism for $k \leq n$, and
2. $\pi_k(Q) = 0$ for $k \geq n$.

We wish to prove that for any space X and any n, the Postnikov approximation (also known as **Postnikov sections**) exists. We will also show that the map $X \to Q$ is unique up to homotopy equivalence.

Theorem 139 *For any space X and any $n \geq 1$, there is a Postnikov approximation $p : X \to Q$.*

Problem 13.25 Prove Theorem 139 by inductively constructing a sequence of spaces Q_j and maps $X \to Q_j$ (with $j \geq n$) such that

1. $\pi_k(X) \to \pi_k(Q_j)$ is an isomorphism for $k \leq n$,
2. $\pi_k(Q_j) = 0$ for $n < k \leq j$, and
3. Q_{j+1} is obtained from Q_j by attaching $(j+1)$-dimensional cells.

I'll even start the inductive construction for you by setting $Q_n = X$.

Now we study the naturality properties of Postnikov sections, which leads, naturally, to some conclusions about their uniqueness.

Problem 13.26 Let $X \to Q$ be any map with $\pi_k(Q) = 0$ for $k \geq n$, and let $X \to P_n(X)$ be the particular Postnikov approximation you constructed in Problem 13.25.

(a) Show that there is a map $P_n(X) \to Q$ making the diagram

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
P_n(X) \arrow[rr] & & Q.
\end{tikzcd}$$

strictly commutative.

(b) Show that Postnikov approximations are unique up to weak homotopy equivalence. For what spaces are they unique up to genuine homotopy equivalence?

Since the n^{th} Postnikov section of X is unique up to (weak) homotopy type, we adopt the notation $P_n(X)$ for it.

Problem 13.27 Show that there are maps $P_{n+1}(X) \to P_n(X)$ making the diagram

$$\begin{tikzcd}
 & X \arrow[dl] \arrow[dr] & \\
P_n(X) \arrow[rr] & & P_{n+1}(X)
\end{tikzcd}$$

strictly commutative. What can you say about the homotopy groups of the fiber of $P_{n+1}(X) \to P_n(X)$?

13.5 Eilenberg-Mac Lane Spaces

One very important class of spaces to which our simple obstruction theory applies is those spaces which have only one nonzero homotopy group. Such spaces are called Eilenberg-Mac Lane spaces.

Let G be a group, and let $n \in \mathbb{N}$. A space L is called an **Eilenberg-Mac Lane space of type** (G, n) if its homotopy groups are given by

$$\pi_k(L) \cong \begin{cases} G & \text{if } k = n \\ 0 & \text{otherwise.} \end{cases}$$

Notice that we used $\cong$ instead of $=$ in this definition. This is an important point, so let's explore it. The group G is a certain set with a multiplication rule. For example, it might be a set of permutations (which are bijective functions from a set to itself) or it might be a set of cosets, etc. The group $\pi_n(L)$ is a set of homotopy classes of maps from a sphere into L. Unless the stars align perfectly, it is essentially impossible for $\pi_n(L)$ to be *equal* to G. Instead, when we say that L is an Eilenberg-Mac Lane space of type (G, n), we mean that the only nonzero homotopy group of L is $\pi_n(L)$ and we have in mind an explicit group isomorphism $\pi_n(L) \to G$. Most of the time, these technical considerations can be safely ignored, but it is occasionally crucial to keep complete control of the situation.

Our goal in this section is to determine some crucial properties of Eilenberg-Mac Lane spaces. Notice that Eilenberg-Mac Lane spaces are defined in terms of maps *into* them, so they should be thought of as target-type spaces.

Beware! Even though we have defined Eilenberg-Mac Lane spaces, we do not yet know that they exist!

PROBLEM 13.28

(a) Show that if there is an $(n-1)$-connected space X with $\pi_n(X) \cong G$, then there is an Eilenberg-Mac Lane space of type (G, n).

(b) Show that S^1 is an Eilenberg-Mac Lane space of type $(\mathbb{Z}, 1)$.

(c) Show that for every group G, there are Eilenberg-Mac Lane spaces of type $(G, 0)$ and $(G, 1)$.

(d) Show that there are groups G for which there is no Eilenberg-Mac Lane space of type (G, n) with $n > 1$.

An Eilenberg-Mac Lane space of type $(G, 1)$ is also called a **classifying space** for the group G, and denoted BG.

Proposition 140 *Let G be a group, and let L be an Eilenberg-Mac Lane space of type (G, n). Then for any $(n-1)$-connected CW complex X, the* *map*

$$\phi : [X, L] \to \hom(\pi_n(X), G) \qquad \text{given by} \qquad \phi(f) = f_*$$

is bijective.

To prove the surjectivity of ϕ in the case $n = 1$, we need to know that $\pi_1(\bigvee_{\mathcal{J}} S^1)$ is the free group $F(\mathcal{J})$; you proved this in the section on the Seifert-Van Kampen theorem. For $n > 1$ we need the analogous formula

$$\pi_n(\bigvee_{\mathcal{J}} S^n) \cong \bigoplus_{\mathcal{J}} \mathbb{Z} \cdot [\text{in}_j].$$

PROBLEM 13.29 Show that if $\pi_n(S^n) \cong \mathbb{Z} \cdot [\text{id}]$, then $\pi_n(\bigvee_{\mathcal{J}} S^n) \cong \bigoplus_{\mathcal{J}} \mathbb{Z} \cdot [\text{in}_j]$.

We will prove that $\pi_n(S^n) \cong \mathbb{Z} \cdot [\text{id}]$ for all n in the next chapter. The skeptical reader should consider Proposition 140 proved entirely in the case $n = 1$, and reserve judgement on the higher dimensions until the end of the next chapter.

PROBLEM 13.30 For the purposes of this problem, either assume that $\pi_n(S^n) \cong \mathbb{Z} \cdot [\text{id}]$ or solve it only in the case $n = 1$.

(a) Explain why it is enough to prove Proposition 140 in the special case that X has dimension at most $n + 1$.

(b) Then use Proposition 137 to show that ϕ is injective.

(c) Show that an $(n - 1)$-connected and $(n + 1)$-dimensional CW complex X sits in a cofiber sequence $\bigvee_{i \in \mathcal{I}} S^n \xrightarrow{\alpha} \bigvee_{j \in \mathcal{J}} S^n \xrightarrow{q} X$, and that the induced map $q_* : \pi_n(\bigvee_{\mathcal{J}} S^n) \to \pi_n(X)$ is surjective.

(d) Let $\pi_n(X) = H$ and let $h : H \to G$ be any group homomorphism. Show that there is a map $\beta : \bigvee_{j \in \mathcal{J}} S^1 \to K(G, 1)$ whose induced map $\pi_n(\bigvee_{\mathcal{J}} S^n) \to G$ is $h \circ q_*$.

(e) Show that ϕ is surjective.

Corollary 141 *If K be an Eilenberg-Mac Lane space of type (G, n) and let L be an Eilenberg-Mac Lane space of type (H, n). If $G \cong H$, then $K \simeq L$.*

PROBLEM 13.31 Use Proposition 140 to prove Corollary 141.

Since our proof of Corollary 141 uses Proposition 140, we should consider it proved for $n = 1$, and provisional for $n > 1$. On the other hand, since we will eventually verify it in full generality, we will write $K(G, n)$ to denote an Eilenberg-Mac Lane space of type (G, n). Remember that we have a particular isomorphism $\pi_n(K(G, n)) \cong G$ in mind.

Proposition 142 *If $n > 0$ then $\Omega K(G, n) \simeq K(G, n - 1)$.*

PROBLEM 13.32 Prove Proposition 142.

Proposition 142 implies that if $K(G,n)$ exists, then $K(G,n-1)$ is an associative H-space. But what about $K(G,n)$? Must it be an H-space?

PROBLEM 13.33 Let G be a group with multiplication $\mu : G \times G \to G$, and suppose $K(G,n)$ exists.

(a) Show that G is abelian if and only if μ is a homomorphism.

(b) Show that if G is abelian, then the homomorphism μ induces a map $m : K(G,n) \times K(G,n) \to K(G,n)$ which is an H-space multiplication.

(c) Show that in the situation of part (b), the multiplication m is homotopy associative.

(d) Show that if G is not abelian, then G is *not* an H-space.

Chapter 14

Suspension

We will show that the suspension map

$$\sigma : X \to \Omega\Sigma X$$

adjoint to the identity $\Sigma X \to \Sigma X$ can be constructed in a domain-type way, called the James Construction. The James construction makes it possible for us to understand the suspension operation $[X, Y] \to [\Sigma X, \Sigma Y]$, leading us to the Freudnthal Suspension Theorem and a variety of computations.

14.1 The Suspension Map

Make more exercises/problems out of this.

The suspension functor defines a natural transformation $\Sigma : [\,?\,,\,?\,] \to [\Sigma\,?\,, \Sigma\,?\,]$. We also have a natural isomorphism

$$\alpha : [\Sigma A, \Sigma X] \to [A, \Omega\Sigma X]$$

(coming from the exponential law) for any two spaces A and X. Putting these together and writing $S = \alpha \circ \Sigma$, we have a commutative diagram of functors and natural transformations

$$\begin{tikzcd}
{[A, X]} \arrow[r, "\Sigma"] \arrow[d, equal] & {[\Sigma A, \Sigma X]} \arrow[d, "\alpha", "\cong"'] \\
{[A, X]} \arrow[r, "S"] & {[A, \Omega\Sigma X].}
\end{tikzcd}$$

By the Yoneda Lemma (Exerecise **??**), the map S is in fact σ_* for some map $\sigma : X \to \Omega\Sigma X$. This diagram shows that the study of the natural transformation Σ is equivalent to the study of σ_*.

Problem 14.1 Show that $\sigma(x) = \omega_x$, where $\omega_x : t \mapsto [x,t]$.

14.2 Moore Paths and Loops

We begin by developing the theory of measured paths. The space of **Moore paths** (also called **measured paths**) on a space X is

$$\mathrm{Path}_M(X) = \big\{(\omega, a) \,|\, \omega|_{[a,\infty)} \text{ is constant}\big\} \subseteq \mathrm{map}\left([0,\infty), X\right) \times [0,\infty)$$

The **endpoint** of the Moore path (ω, a) is the point $\omega(a) \in X$. If (ω, a) and (τ, b) are Moore paths in X with the endpoint of (ω, a) equal to the start point of (τ, b), (i.e., with $\omega(a) = \tau(0)$) then we may **concatenate** them by setting $(\omega, a) \star (\tau, b) = (\omega \star \tau, a+b)$, where

$$\alpha \star \beta(t) = \begin{cases} \alpha(t) & \text{if } t \le a \\ \beta(t-a) & \text{if } t \ge a. \end{cases}$$

Exercise 14.2

(a) Show how to consider X^I as a subspace of $\mathcal{P}_M(X)$. How are the concatenations $\omega * \tau$ and $(\omega, a) \star (\tau, b)$ related?

(b) Show that $\mathrm{map}([0,\infty), X) \simeq X$.

(c) Show that $\mathrm{map}_*([0,\infty), X) \simeq *$.

For $X \in \mathcal{T}_*$, the **Moore path space** on X is

$$\mathcal{P}_M(X) = \{(\omega, t) \in \mathrm{Path}_M(X) \,|\, \omega(0) = *\}.$$

There is an evaluation map $@ : \mathcal{P}_M(X) \to X$ given by $(\omega, t) \mapsto \omega(t)$.

Problem 14.3 Show that @ is a fibration.

Hint Write down a lifting function.

The fiber of @ is $\Omega_M X$, the space of **Moore loops**.

Problem 14.4 Show that the rule $(\omega, t) \mapsto \boxed{s \mapsto \omega(st)}$ defines a homotopy equivalence $\mathrm{Path}_M(X) \to X^I$. Show that it restricts to homotopy equivalences

$$\mathcal{P}_M(X) \to \mathcal{P}(X) \qquad \text{and} \qquad \Omega_M(X) \to \Omega X.$$

Conclude that the maps $\mathcal{P}_M(X) \to X$ and $\mathcal{P}(X) \to X$ are homotopy equivalent.

Problem 14.5 Show that $\Omega_M X$ is a topological monoid and that the homotopy equivalence $\Omega_M X \to \Omega X$ is an H-map.

Composing Infinite Collections of Homotopies. We have discussed this already, but now we have a different point of view available. A homotopy $H : A \times I \to X$ is adjoint to a map $\widehat{H} : A \to \mathrm{Path}(X)$. More generally, a homotopy indexed on the interval $[0, a]$ corresponds to an eventually constant homotopy $\widehat{H} : A \to \mathrm{Path}_M(X)$, given explicitly by

$$\widehat{H}(x) = \left(H|_{\{x\}\times[0,a]}, a\right).$$

This is nice and compatible with rigid concatenation of homotopies.

Let $H_1, H_2, \ldots$ be an infinite collection of homotopies. Their infinite concatenation defines a function $H : X \times [0, \infty) \to Y$, which has an adjoint $\widetilde{H} : X \to \mathrm{map}([0, \infty), Y)$. If there is a continuous function $z : X \to [0, \infty)$ such that $H_n|_{x\times I}$ is constant if $n \geq z(x)$, Then we may define

$$\widehat{H} : X \to \mathrm{Path}_M(X)$$

by the rule

$$\widehat{H}(x) = (\widetilde{H}(x), z(x)).$$

This is a homotopy from $H_1|_{X\times 0}$ to … *what?*

Define $X(n) = z^{-1}([0, n]) \subseteq X$. Then we have maps $g_n : X(n) \to Y$; specifically, $g_n = H|_{X(n)\times n}$, the constant part of the homotopy. These maps extend one another giving a map from the telescope diagram

$$X(0) \to X(1) \to \cdots \to X(n) \to X(n+1) \to \cdots$$

into Y. If X is the colimit of this telescope diagram, then this gives a map $X \to Y$, which we interpret as the end of the homotopy.

Problem 14.6 Show that the composite the lift H exists in the diagram

$$\begin{tikzcd}
 & & Y^I \arrow[d] \\
X \arrow[r] \arrow[urr, dotted, "H"] & \mathcal{P}_M(Y) \arrow[r] & Y^{[0,1)}
\end{tikzcd}$$

and H is a homotopy from f to g.

Exercise 14.7 Compare with our earlier approach.

14.3 The James Construction

The James Construction gives an explicit domain-type construction of the suspension map $X \to \Omega\Sigma X$.

Definition of the James Construction. We let X be a CW complex with a single 0-cell $*$, and consider the n-fold products X^n for $n \geq 0$. We consider $X^n \subseteq X^{n+1}$ in the usual way: by identifying the point $(x_1, x_2, \ldots, x_n) \in X^n$ with the point $(x_1, x_2, \ldots, x_n, *) \in X^{n+1}$.

The union of all of the X^n is X^∞, the set of all **finite** sequences of points of X. Formally, X^∞ is the colimit of the diagram

$$X^0 \to X^1 \to X^2 \to \cdots \to X^n \to X^{n+1} \to \cdots.$$

And since all the inclusions $X^n \to X^{n+1}$ are cofibrations, the colimit is also a homotopy colimit. This is sometimes known as the **weak infinite product**.

Now we define an equivalence relation on X^∞: two points $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ are equivalent if and only if, after deleting all entries $*$, they are the exactly the same. For example, if $*, a, b \in X$, then

$$(a, b, *, *, a, *, b) \sim (*, *, *, a, *, b, a, *, *, *, *, b, *, *) \quad \text{in} \quad X^\infty.$$

We let $J(X)$ denote the set of equivalence classes of points in X^∞; we of course have a quotient map $q : X^\infty \to J(X)$.

Exercise 14.8 Show that J is a functor. Show that if X is a CW complex, then so is $J(X)$.

Recall that a **monoid** is an algebraic gadget that satisfies all the rules for a group, except possibly the rule that every element has an inverse – I leave it to you to define a **monoid object** and a **monoid homomorphism** in a category $\mathcal{C}$. A topological monoid is automatically an **H-space**, but an H-space need not be a topological monoid, since a monoid must be strictly associative.

Exercise 14.9 If X is a topological monoid, then the functor $[\,?\,, X]$ takes its values in the category of monoids.

Proposition 143 *For any space X, the map*

$$\mu : J(X) \times J(X) \to J(X)$$

given by $\mu((x_1, \ldots, x_n), (y_1, \ldots, y_m)) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ makes $J(X)$ into a topological monoid.

It is often useful to think about $J(X)$ is as the free monoid generated by the set X.

Problem 14.10 Show that any map $f : X \to M$, where M is a topological monoid, extends to a unique monoid map $\phi : J(X) \to M$; and if f is continuous, then so is ϕ.

Filtration of $J(X)$. Inside of $J(X)$ we have the images $J^n(X) = q(X^n)$, which is the set of all elements of $J(X)$ with at most n nontrivial entries.

NOTATION From now on, unless we have several spaces floating around, we'll simply write J for $J(X)$ and J^n for $J^n(X)$.

These spaces are related to one another by the maps

$$\mu : X \times J^n \to J^{n+1} \quad \text{given by} \quad (y, (x_1, \ldots, x_n)) \mapsto (y, x_1, \ldots, x_n).$$

If the multiple use of μ bothers you, call this one μ_n ($n = \infty$ allowed). The inclusion map $J^0 \to J^1$ is simply the inclusion $* \hookrightarrow X$, which is a cofibration.

EXERCISE 14.11 In the case $n = \infty$, we get $\mu : X \times J \to J$. Show that the map $\mu_* : \pi_k(X \times J) \to \pi_k(J)$ is given by $\mu_*(\alpha, \beta) = j_*(\alpha) + \beta$, where $j : X \hookrightarrow J$.

PROBLEM 14.12

(a) Show that there is a categorical pushout square

$$\begin{tikzcd} X \times J^n \cup * \times J^{n+1} \arrow[r, "i"] \arrow[d, "\mu \cup \mathrm{pr}_2"'] & X \times J^{n+1} \arrow[d, "\mu"] \\ J^{n+1} \arrow[r, hook] & J^{n+2}. \end{tikzcd}$$

(b) Prove that each inclusion $J^n \to J^{n+1}$ is a cofibration.

HINT Work by induction using the square you just proved is a pushout, using Theorem 41.

(c) Conclude that these squares are all homotopy pushout squares.

(d) Show that $J^n(X)/J^{n-1}(X) \cong X^{\wedge n}$.

Identifying the Suspension Map. The suspension map σ corresponds, via the inclusion $\Omega\Sigma X \hookrightarrow \Omega_M \Sigma X$ to the map $\sigma_M : x \mapsto (\omega_x, 1) \in \mathcal{P}_M(X)$. Since $J(X)$ is a free monoid on X, and $\Omega_M \Sigma X$ is a monoid, there is a unique monoid map $e : J(X) \to \Omega_M \Sigma X$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "j"'] \arrow[d, "\sigma_M"] \arrow[dr, "\sigma"] & \\ J(X) \arrow[r, "e"] & \Omega_M \Sigma X & \Omega\Sigma X \arrow[l, "\simeq"] \end{tikzcd}$$

is strictly commutative.

EXERCISE 14.13 What is $e([x_1, x_2, \ldots, x_n])$?

Now we can state and prove our theorem identifying the inclusion $X \hookrightarrow J(X)$.

Theorem 144 *For every CW complex X, the map e is a homotopy equivalence and the diagram*

$$\begin{tikzcd}
X \arrow[r, "j"] \arrow[d, equal] & J(X) \arrow[d, "e", "\simeq"'] \\
X \arrow[r, "\sigma_M"] & \Omega_M \Sigma X
\end{tikzcd}$$

is strictly commutative.

The proof of our theorem amounts to a detailed study of the commtuative cube

$$\begin{tikzcd}
X \times J \arrow[rr, "\mathrm{p}_2"] \arrow[dr, "\mu"] \arrow[dd, "\mathrm{pr}_1"'] & & CX \times J \arrow[dr] \arrow[dd, "\mathrm{pr}_1"' near end] & \\
& J \arrow[rr, "\xi" near start] \arrow[dd] & & T \arrow[dd, "q"] \\
X \arrow[rr] \arrow[dr] & & CX \arrow[dr] & \\
& * \arrow[rr] & & \Sigma X.
\end{tikzcd}$$

in which the back and left faces are strong homotopy pullback, and top and bottom are categorical and homotopy pushouts.

Problem 14.14

(a) Show that the back two vertical squares are strong homotopy pullback squares.

(b) Show that q is a weak fibration and that the front two squares are categorical pullback squares.

(c) Show that the front two squares are strong homotopy pullback squares.

Now let us study the space T in greater detail. By construction, we have $T = (CX \times J)/\sim$ where

$$([x,0],[x_1,\ldots,x_n]) \sim (*,[x,x_1,\ldots,x_n]).$$

Under this identification, the map q is given by

$$q(([x,t],[x_1,\ldots,x_n]) = [x,t].$$

Each point in T, except $*$, has a unique expression in the form $([x,t],[x_1,\ldots,x_n])$ with $t<1$. Also the map $\xi : J \to T$ is given by

$$\xi([x_1,\ldots,x_n]) = (*,[x_1,\ldots,x_n]).$$

Let $T_n = J_n \times J/\sim \subseteq T$. The map $H : T \times I \to T$ given by

$$H(([x,t],[x_1,\ldots,x_n]),s) = ([x,(1-s)t+s],[x_1,\ldots,x_n])$$

deforms the space T in such a way that T_n is compressed, within itself, into T_{n-1} for each n. That is, H is a homotopy from id_T to the map $d : T \to T$ given by the formula

$$d : ([x_0,t],[x_1,\ldots,x_n]) \mapsto ([x_1,0],[x_2,\ldots,x_n]).$$

Problem 14.15 Show that d^∞, the infinite composite of d with itself, is defined and constant. Show that infinite composition of H with itself is a homotopy $H^\infty : \mathrm{id}_T \simeq d^\infty$.

To finish our proof, we study the front square of our cube in more detail. Since $T \simeq *$, the map $T \to \Sigma X$ factors through the path fibration $\mathcal{P}_M\Sigma X \to \Sigma X$, and we obtain the diagram

$$\begin{tikzcd}
J \arrow[r, "\xi"] \arrow[d, "f"'] & T \arrow[d, "u"] \\
\Omega_M\Sigma X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & \mathcal{P}_M(\Sigma X) \arrow[d, "@_1"] \\
* \arrow[r] & \Sigma X.
\end{tikzcd}$$

Since the front square is a strong homotopy pullback, the map f is a homotopy equivalence. But what is this map?

Problem 14.16 Show that the induced map $J \to \Omega_M\Sigma X$ is given by

$$[x_1,\ldots,x_n] \mapsto (\omega_{x_1} \star \cdots \star \omega_{x_n}, n).$$

Explain why this proves Theorem 144.

Hint Since J is the colimit of the $J_n \subseteq J$, we only need to determine the restriction of the homotopy to J_n.

14.4 The James Splitting and the Hilton-Milnor Theorem

The suspension of the James Construction is easily analyzed, and this gives valuable information about $\Sigma\Omega\Sigma X$. This is used, for example, in the proof our version of the Hilton-Milnor Theorem, which describes the loop space of a wedge.

The James Splitting. This problem gives an explicit formula for $\Sigma\Omega\Sigma X$; it is called the **James splitting** of the loop space of a suspension. Historically, it was derived as a consequence of the work you will do in the next chapter.

Problem 14.17

(a) Show that the suspension of the inclusion $J^{n-1}(X) \hookrightarrow J^n(X)$ can be identified as in the diagram

$$\begin{tikzcd}
\Sigma J^{n-1}(X) \arrow[r] \arrow[d, equal] & \Sigma J^n(X) \arrow[d, "\simeq"] \\
\Sigma J^{n-1}(X) \arrow[r, hook] & \Sigma J^{n-1}(X) \vee \Sigma X^{\wedge n}.
\end{tikzcd}$$

(b) Show that $\Sigma\Omega\Sigma X \simeq \Sigma\left(\bigvee_{n\geq 0} X^{\wedge n}\right)$.

The Hilton-Milnor Theorem. The wedge of two (or more) spaces is fundamentally a domain-type object, and so computing the homotopy groups of a wedge is a fundamentally difficult and interesting problem. In this section, we work out a way to approach this problem.

We have seen that some apparently indivisible spaces, like ΩS^n, actually break into smaller, more easily understood spaces when they are suspended. This is quite useful if we are interested in the cohomology of the space (e.g., in Problem 17.8(d)). But here we are concerned with homotopy groups of $X \vee Y$, so we instead look at $\Omega(X \vee Y)$, and look for splittings of this loop space into a *product* of smaller, more easily understood spaces.

Problem 14.18 Let

$$\begin{tikzcd}
F \arrow[r, "i"'] & E \arrow[r, "p"'] & B \arrow[l, bend right, dotted, "\sigma"']
\end{tikzcd}$$

be a fibration sequence, and assume that there is a map $\sigma : B \to E$ such that $p \circ \sigma \simeq 1_B$ (such a map σ is called a **homotopy section** of the map p).

(a) Show that the fibration $\Omega P : \Omega E \to \Omega B$ also has a section.

(b) Use the maps Ωi and Ωp and the fact that ΩE is an H-space (i.e., you can multiply elements in ΩE together, and there is an identity element) to construct a map $e : \Omega F \times \Omega B \to \Omega E$.

(c) Show that the diagram

$$\begin{tikzcd}
\Omega F \arrow[r, "\mathrm{in}_1"] \arrow[d, "1_F"'] & \Omega F \times \Omega B \arrow[r, "\mathrm{pr}_2"] \arrow[d, "e"] & \Omega B \arrow[d, "1_{\Omega B}"] \\
\Omega F \arrow[r, "\Omega i"] & \Omega E \arrow[r, "\Omega p"] & \Omega B
\end{tikzcd}$$

commutes; why is it true that $e \circ \mathrm{in}_1 \simeq \Omega i$?

(d) Show that e is a homotopy equivalence, so that $\Omega E \simeq \Omega F \times \Omega B$. This is known as a **splitting** of the loop space ΩE.

Hint The top row of the diagram is also a fibration sequence.

(e) Show that, for every k, $\pi_k(E) \cong \pi_k(F) \oplus \pi_k(B)$.

Here is another situation in which loop spaces split after looping.

Problem 14.19 Let

$$F \xrightarrow{i} E \xrightarrow{p} B$$

be a fibration sequence, and assume that $i : F \to E$ is trivial.

(a) Show that $\partial : \Omega B \to F$ has a homotopy section.

Hint Compare with Problem 8.8.

(b) Show that $\Omega B \simeq F \times \Omega E$.

(c) Let $S^n \to S^m \to B$ be a fibration sequence with $n < m$. prove that $\Omega B \simeq S^n \times \Omega S^m$.

The following problem is the bulk of the work needed to prove the Hilton-Milnor Theorem, which concerns the homotopy groups of a wedge of two spaces. This approach to the Hilton-Milnor Theorem is due to Gray [`?Gray`].

Problem 14.20 Let A and B be any two spaces, and consider the collapse map

$$f : A \vee B \to A.$$

Let F be the homotopy fiber of f.

(a) Show that f has a homotopy section.

(b) Use the cube theorem to determine the homotopy type of F.

(c) Show that $\Omega(A \vee B) \simeq \Omega A \times F$.

Let's study the half-smash products which involve suspensions.

Exercise 14.21

(a) Show that $X \rtimes Y = X \wedge Y_+$.

(b) Show that $S^1 \rtimes Y_+ \simeq \Sigma(Y \vee S^1)$.

Hint Draw a picture!

(c) Conclude that $\Sigma X \rtimes Y = X \wedge (S^1 \wedge Y_+) = \Sigma X \vee \Sigma(X \wedge Y)$

(d) Conclude that $S^m \rtimes S^n \simeq S^m \vee S^{n+m}$.

Now you can apply Problem 14.20 to a wedge of spheres.

Exercise 14.22 Let $m \leq n$

(a) Apply Problem 14.20 to $S^m \vee S^n$.

(b) Use the James Splitting for ΩS^n to rewrite your answer in terms of the loop space of another wedge of spheres.

(c) Repeat, always collapsing to the smallest sphere. What happens to the connectivity of the confusing loop space piece as you repeat the process?

(d) Argue that $\Omega(S^m \vee S^n)$ is homotopy equivalent to a big product of loop spaces of spheres of various dimensions.

The list of spheres that show up in this splitting of $\Omega(S^m \vee S^n)$ has actually been worked out semi-explicitly by Peter Hilton. They may be put in one-to-one correspondence with a basis for a free graded Lie algebra.

14.5 The Freudenthal Suspension Theorem

Theorem 144 gives, up to homotopy equivalence of maps, an explicit description of CW structure of the suspension map σ.

Lemma 145 *If Y is $(k-1)$-connected, then*

$$J^{n+1}(Y) = J^n(Y) \cup (\text{cells of dimension } \geq k(n+1)).$$

In particular, the map $\sigma : Y \hookrightarrow \Omega\Sigma Y$ is a $(2n-1)$-equivalence.

Problem 14.23 Prove Lemma 145.

This implies the *Freudenthal Suspension Theorem.*

Theorem 146 *Let Y be an $(n-1)$-connected space. Then*

(a) if $\dim(X) < 2n-1$, then $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is an isomorphism, and

(b) if $\dim(X) = 2n-1$, then $\Sigma : [X, Y] \to [\Sigma X, \Sigma Y]$ is surjective.

The following special case is particularly important.

Corollary 147 *Let Y be an $(n-1)$-connected space. Then*

(a) *$\Sigma : \pi_k(Y) \to \pi_{k+1}(\Sigma Y)$ is an isomorphism for $k < 2n-1$, and*

(b) *$\Sigma : \pi_{2n-1}(Y) \to \pi_{2n}(\Sigma Y)$ is onto.*

PROBLEM 14.24 Prove Theorem 146 and Corollary 147.

We have already shown, using cellular approximations, that $\operatorname{conn}(\Sigma X) \geq \operatorname{conn}(X) + 1$. Now we are prepared to show that this is an equality for simply-connected spaces.

PROBLEM 14.25 Show that if X is simply-connected then $\operatorname{conn}(\Sigma X) = \operatorname{conn}(X) + 1$. Does your argument work for path-connected spaces X that are not simply-connected?

Before getting to the non-simply-connected case, we take a few detours.

Low-Dimensional Homotopy Groups of Spheres. We now determine the groups $\pi_k(S^n)$ for $k \leq n$. This is arguably the most important calculation in all of homotopy theory. All other calculations are ultimately founded – at least in part – on this theorem.

Theorem 148 *For all $n > 0$, the suspension map $\Sigma : \pi_n(S^n) \to \pi_{n+1}(S^{n+1})$ is an isomorphism, and hence*

$$\pi_n(S^n) = \begin{cases} \mathbb{Z} \cdot [\mathrm{id}_{S^n}] & \text{if } k = n \\ 0 & \text{if } k < n. \end{cases}$$

The groups $\pi_k(S^n)$ for $k > n$ are much more difficult to compute; indeed, they have only been computed one at a time, and our knowledge of them peters out around $k = n + 70$. (There are also periodic phenomena and global properties that are known, so it's not a total mystery.)

We've already know Theorem 148 in the case $n = 1$.

PROBLEM 14.26

(a) Suppose Y is a strictly associative H-space. Show that the suspension map $\sigma : Y \to \Omega\Sigma Y$ has a left homotopy inverse τ.

(b) Show that $\Sigma : \pi_1(S^1) \to \pi_2(S^2)$ is an isomorphism.

(c) Prove Theorem 148.

We may now use Proposition 140 without reservation.

Finally we have a functor that can distinguish between S^n and D^{n+1}, and this is good news because it shows that our intuition that $S^n \not\simeq *$ is accurate.

EXERCISE 14.27 Generalize and prove Exercise **??**.

Eilenberg-Mac Lane Spaces. We can now show that Eilenberg-Mac Lane spaces for abelian groups exist.

PROBLEM 14.28

(a) Show that there is an Eilenberg-Mac Lane space of type $K(\mathbb{Z}, n)$ for all $n \geq 1$.

(b) Show that for any abelian group G and any $n \geq 1$, there is an Eilenberg-Mac Lane space of type (G, n).

HINT Start with a short exact sequence $0 \to \bigoplus_{\mathcal{I}} \mathbb{Z} \to \bigoplus_{\mathcal{J}} \mathbb{Z} \to G \to 0$; then use Proposition 140.

(c) Show that there is a $K(G, n)$ which is a strictly associative H-space.

HINT Use the Moore loop space.

PROBLEM 14.29 Show that $K(\mathbb{Z}, n)$ can be built from S^n by attaching cells of dimension $\geq n+2$. Thus the inclusion $S^n \hookrightarrow K(\mathbb{Z}, n)$ induces an isomorphism $\mathbb{Z} \to \pi_n(K(\mathbb{Z}, n))$, and this is the explicit isomorphism we have use when it is needed.

Suspension in Dimension 1. Recall that the **commutator** of two elements $x, y \in G$ is the element $[x, y] = x^{-1}y^{-1}xy \in G$. The **commutator subgroup** of a group G is the subgroup of $G' \subseteq G$ generated by all of the commutators, and the **abelianization** of G is the quotient map $G \to G/G'$ (it is the 'largest abelian quotient' of G). We also use 'abelianization' to describe any surjective homomorphism $G \to H$ whose kernel is G'.

Proposition 149 *The suspension map $\Sigma : \pi_1(X) \to \pi_2(\Sigma X)$ is abelianization.*

PROBLEM 14.30 Prove Proposition 149.

HINT Let A be the abelianization of $\pi_1(X)$ and consider maps $X \to K(A, 1)$.

We now deduce that suspension usually, *but not always*, increases connectivity by one.

Theorem 150 *Let $f : X \to Y$, and assume that $\pi_1(X)$ and $\pi_1(Y)$ are both abelian. Then*

(a) $\mathrm{conn}(\Sigma X) = \mathrm{conn}(X) + 1$, *and*

(b) $\mathrm{conn}(\Sigma f) = \mathrm{conn}(f) + 1$.

PROBLEM 14.31 Prove Theorem 150

EXERCISE 14.32 Show that if $\mathrm{Ab}(\pi_1(X)) = 0$, then $\mathrm{conn}(\Sigma X) \geq 2 > \mathrm{conn}(X)+1$. For each part of Theorem 150, find weaker hypotheses that lead to the same conclusions.

Chapter 15

Comparing Pushouts and Pullbacks

In Chapter **??** we used cellular approximations to estimate the connectivity (target-type) of spaces constructed by domain-type methods. In this chapter we go even deeper into the domain/target mix by comparing homotopy pushout squares and homotopy pullback squares. If the square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

is a homotopy pullback square, then we can form the homotopy pushout P of $C \leftarrow A \rightarrow B$ and consider the induced comparison maps $P \rightarrow D$. We determine the homotopy fiber of these maps, and thereby deduce the connectivity of the map. We pay particular attention to the special case in which $C = *$, so the square corresponds to a fiber sequence, and the comparison map is the canonical map $E/F \rightarrow B$.

As you have shown in Problem **??**, the fiber of the comparison map $A \rightarrow P$ is the same as the **iterated fiber** obtained by forming the homotopy fiber of the induced maps between homotopy fibers of the columns (or rows) of the square.

We also consider the dual problem. If the square is a homotopy pushout square, we may form the homotopy pullback Q of $C \rightarrow D \leftarrow B$ and study the comparison maps $A \rightarrow Q$. In the special case $C = *$, we have a cofiber sequence $A \rightarrow B \rightarrow D$ and, if F is the homotopy fiber of $B \rightarrow D$, a comparison map $A \rightarrow F$. We identify the *suspension* of this map, and estimate its connectivity. This preliminary result is the foundation of our general result, which is proved by CW induction.

15.1 Comparing Homotopy Pullbacks to Homotopy Pushouts

Start with a homotopy pullback square

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPB}"] & X \arrow[d] \\ Y \arrow[r] & Z, \end{tikzcd}$$

and form the (homotopy) pushout, D, of the diagram $Y \leftarrow W \rightarrow X$, and call it D. Then we get a (nonunique) comparison map $\xi : D \to Z$, and we want to estimate – or determine – its connectivity. This is complicated by the fact that ξ is not uniquely defined, even up to homotopy.

Exercise 15.1

(a) Show that ξ is well defined up to homotopy equivalence of maps, and conclude that the homotopy type of the fiber is also well defined. Show that any two choices of ξ have the same connectivity.

(b) Clearly explain the sense in which the formation of the homotopy fiber of $D \to Z$ is functorial.

(c) Dualize the previous two parts.

Now we have honest questions: what is the (common) homotopy fiber of the induced maps $\xi : D \to Z$? And how highly connected is it?

We now lay the groundwork for our inductive approach to the problem of estimating how close a homotopy pushout square is to being a homotopy pullback square. We need to relate the homotopy fibers of the squares in a composition.

Problem 15.2 Consider the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "(I)"] & B \arrow[r] \arrow[d] \arrow[dr, phantom, "(II)"] & C \arrow[d] \\ X \arrow[r] & Y \arrow[r] & Z. \end{tikzcd}$$

Let $P_{(I)}$ be the homotopy pullback of square (I), let $\xi_{(I)} : A \to P_{(I)}$ be the comparison map, and write $F_{(I)}$ for its homotopy fiber. Similarly, let $\xi_{(II)} : B \to P_{(II)}$ be the comparison map for square (II), with homotopy fiber $F_{(II)}$. Finally, let

$\zeta : A \to Q$ be the comparison map for the composite square and write G for its fiber.

(a) Show that there is a factorization

$$\begin{tikzcd} A \arrow[rr, "\zeta"] \arrow[dr, "\xi_{(I)}"'] & & Q \\ & P_{(I)} \arrow[ur, "\theta"'] & \end{tikzcd}$$

(b) Show that there is a fiber sequence $F_{(I)} \to G \to F_{(II)}$.

(c) Discuss the relationships between the connectivities of the three iterated fibers.

15.2 From Homotopy Pullback to Homotopy Pushout

Here we resolve our first problem: if a square is a homotopy pullback square, then to what extent can it be treated also as a homotopy pushout square?

Theorem 151 *If $X \to Z$ is an n-equivalence and $Y \to Z$ is an m-equivalence, then $D \to Z$ is an $(n+m)$-equivalence.*

PROBLEM 15.3 Pull back from the fibration $\mathcal{P}Z \to Z$, to construct a cube whose base is our homotopy pushout square.

(a) Determine the spaces and maps in the top of the cube.

(b) Identify the homotopy type of the fiber F of $\xi : D \to Z$. Show that your identification is functorial.

(c) Prove Theorem 151.

Ganea's Fiber-Cofiber Construction. The most important special case of Theorem 151 is when the homotopy pullback square we start with corresponds to a fibration sequence $F \to E \to B$. Given $F \to E \to B$ we construct the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[d] \arrow[ddr, bend left] & \\ * \arrow[r] \arrow[drr, bend right] & E/F \arrow[dr, dashed, "g"] & \\ & & B \end{tikzcd}$$

The map $g : E/F \to B$ constructed in this way is known as Ganea's **fiber-cofiber construction** on $p : E \to B$. We can go a step further and convert g to a fibration $G(p) : G(E) \to B$; this is called the **Ganea construction** on p.

PROBLEM 15.4 Determine the homotopy fiber of the map g in terms of the spaces F, E and B, and show that your identification is functorial.

PROBLEM 15.5 Apply the fiber-cofiber construction to the fibration sequence $\Omega X \to \mathcal{P}X \to X$.

(a) Show that the result is naturally homotopy equivalent to the map $\lambda : \Sigma\Omega X \to X$ adjoint to $\mathrm{id}_{\Omega X}$.

HINT You can embed $C\Omega X$ into $\mathcal{P}X$ using the map $(\omega, t) \mapsto \omega_t$, where

$$\omega_t(s) = \begin{cases} \omega(s) & \text{if } s \le t \\ \omega(t) & \text{if } s \ge t. \end{cases}$$

(b) What is the connectivity of λ?

Now we can prove a dual to the Freudenthal Suspension Theorem.

Theorem 152 *Suppose X is an $(n-1)$-connected CW complex and $\pi_k(Y) = 0$ for $k \ge 2n-2$. Then the looping map*

$$\Omega : [X, Y] \to [\Omega X, \Omega Y]$$

is an bijective. If $\pi_k(Y) = 0$ for $k \ge 2n-1$, then Ω is surjective.

PROBLEM 15.6 Prove Theorem 152.

Comparing the Fiber of f to Its Cofiber. We can use Ganea's construction to compare the fiber of a map with its cofiber. Let $f : A \to B$ with cofiber C and fiber F. Then we may construct the diagram

$$\begin{tikzcd}
F \arrow[r] & A \arrow[r] \arrow[d, equal] & A/F \arrow[r] \arrow[d, "g"] & \Sigma F \arrow[d, "\phi"] \\
 & A \arrow[r, "f"] & B \arrow[r] & C.
\end{tikzcd}$$

in which the rows are cofiber sequences.

Theorem 153 *If B is $(b-1)$-connected and f is a $(c-1)$-equivalence, then*

(a) the comparison map $\phi : \Sigma F \to C$ is a $(b+c-1)$-equivalence.

(b) if $b \ge 1$ then C and ΣF have the same connectivity.

Problem 15.7 With the setup above,

(a) What is the connectivity of g?

(b) Prove Theorem 153.

Hint Use Problem 123.

Exercise 15.8 What can you say if $b = 1$?

Theorem 153 very nearly tells us that $\mathrm{conn}(C) = \mathrm{conn}(F) + 1$. What it actually guarantees is that $\mathrm{conn}(C) \geq \mathrm{conn}(\Sigma F) + 1$.

Problem 15.9 What conditions must you impose on the map f and the spaces A and B to be able to conclude $\mathrm{conn}(\Sigma F) = \mathrm{conn}(F) + 1$?

Corollary 154 *if $\pi_1(A)$ is abelian, then $f : A \to B$ is an n-equivalence if and only if Σf is an $(n + 1)$-equivalence.*

Problem 15.10 Prove Corollary 154.

15.3 The Blakers-Massey Cofiber Theorem

Now we come to the dual question: how close is a homotopy pushout square to being a homotopy pullback square? This is a more difficult question because we do not have a dual to the Second Cube Theorem. In this section, we will not pursue the full result, but instead we will concentrate on comparing a given cofiber sequence to a fiber sequence. This has a very satisfying and useful answer: not only can we estimate the connectivity of the fiber of ϕ, we can determine the homotopy type of the map $\Sigma\phi$. We will use this in the next section to derive the full Blakers-Massey Theorem.

If $X \to Y \to Z$ is a cofiber sequence, then it is homotopy equivalent to another sequence of the form

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "q"] & C \end{tikzcd}$$

where $i : A \hookrightarrow B$ is a cofibration, $C = B/A$, and $q : B \to C$ is the canonical quotient map. Let $F = F_q$ be the homotopy fiber of q, which means that we have converted $q : B \to C$ to a fibration $p : \overline{B} \to C$ and $F = p^{-1}(*)$. Thus we have a homotopy commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\xi"'] & B \arrow[r] \arrow[d, "\simeq"'] & C \arrow[d] \\
F \arrow[r] & \overline{B} \arrow[r] \arrow[d] & \overline{B}/F \arrow[d] \\
 & B \arrow[r] & C.
\end{tikzcd}$$

This diagram already contains all that we need to estimate the connectivity of the comparison map $A \to F$.

Problem 15.11

(a) Show that $\Sigma\xi$ has a left homotopy inverse, so that ΣA is a retract of ΣF.

(b) Determine the connectivity of $B/F \to C$, $\Sigma F \to \Sigma A$ and $\Sigma A \to \Sigma F$.

Next we will determine the homotopy type of the map $\Sigma\xi$.

Theorem 155 *With the setup above, the map $\Sigma\xi$ can be naturally identified as in the diagram*

$$\begin{tikzcd}
\Sigma A \arrow[r, "\Sigma\xi"] \arrow[d, equal] & \Sigma F \arrow[d, "\simeq"] \\
\Sigma A \arrow[r, "\mathrm{in}_1"] & \Sigma A \vee (A{*}\Omega C).
\end{tikzcd}$$

If A is $(a-1)$-connected and C is $(c-1)$-connected, then the comparison map ξ is $(a+c-2)$-connected.

Problem 15.12

(a) Using the Second Cube Theorem, show that there is a homotopy pushout square of the form

$$\begin{tikzcd}
A \times \Omega C \arrow[r, "\theta"] \arrow[d, "\mathrm{pr}_2"'] & F \arrow[d] \\
\Omega C \arrow[r] & *.
\end{tikzcd}$$

(b) Show that the comparison map $\xi : A \to F$ factors as

$$\begin{tikzcd}
A \arrow[rr, "\xi"] \arrow[dr] & & F \\
 & A \times \Omega C. \arrow[ur, "\theta"'] &
\end{tikzcd}$$

(c) Prove Theorem 155.

Exercise 15.13 Clearly write down what it means for the identification in Theorem 155 to be natural. Verify that it is natural.

Fiber of Suspension vs. Suspension of Fiber. Let $f : A \to B$, let F_f be its homotopy fiber and let C be its cofiber. Also suspend f and let $F_{\Sigma f}$ be its homotopy fiber. Then we can ask: how do ΣF_f and $F_{\Sigma f}$ compare? To answer the question, we need to construct a map from one to the other; then we can attempt to estimate the connectivity of the map.

The construction of the map is a straightforward combination of the comparison maps we have been studying: it is the vertical composite in the middle of the diagram

$$\begin{tikzcd}
F_f \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & \Sigma F_f \arrow[d, "\xi"] & & \\
A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] \arrow[d, "\zeta"'] & \Sigma A \arrow[r, "\Sigma f"] \arrow[d, equal] & \Sigma B \arrow[d, equal] \\
 & & F_{\Sigma f} \arrow[r] & \Sigma A \arrow[r, "\Sigma f"] & \Sigma B
\end{tikzcd}$$

Proposition 156 *If B is $(b-1)$-connected and f is a $(c-1)$-equivalence, then the comparison map $\zeta \circ \xi : \Sigma F_f \to F_{\Sigma f}$ is a $(b+c-2)$-equivalence.*

Problem 15.14 Prove Proposition 156.

The Blakers-Massey Exact Sequence of a Cofibration. The Blakers-Massey theorem gives us a kind-of-lengthy exact sequence for the homotopy groups of a *cofiber sequence.*

Problem 15.15 Let $A \to B \to C$ be a cofiber sequence, in which A is $(a-1)$-connected and C is $(c-1)$-connected. Show that there is an exact sequence

$$\pi_{a+c-2}(A) \to \pi_{a+c-2}(B) \to \pi_{a+c-2}(C) \to \pi_{a+c-3}(A) \to \cdots$$

$$\cdots \to \pi_1(C) \to \pi_0(A) \to \pi_0(b) \to \pi_0(C).$$

Problem 15.16 What is $\pi_n(M(\mathbb{Z}/a, n))$? Assuming the values for $\pi_k(S^n)$ given in the section on Moore spaces, determine $\pi_{n+1}(M(\mathbb{Z}/z, n))$ (for $n > 2$).

15.4 Blakers-Massey Square Comparison

Now we are equipped to study the extent to which a homotopy pushout square is a homotopy pullback square. Start with

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

form the homotopy pullback P, and the comparison map $\xi : A \to P$. Let F be the homotopy fiber of $A \to B$ and let G be the homotopy fiber of $C \to D$. Then the homotopy fiber of $F \to G$ is the same as the homotopy fiber of $\xi : A \to P$, so we estimate this.

We begin with a special case.

Problem 15.17 Suppose there is cofiber sequence $Z \to A \to B$, where $\operatorname{conn}(A \to B) = \operatorname{conn}(Z) + 1$.

(a) Show that there is a cofiber sequence $Z \to C \to D$.

(b) Identify the suspension $\Sigma F \to \Sigma G$ of the induced map of homotopy fibers.

(c) Show that if F and G are simply-connected, then

$$\operatorname{conn}(A \to P) \geq \operatorname{conn}(A \to B) + \operatorname{conn}(A \to C) - 2.$$

Check for fencepost error.

Thus, the best we can hope for in general is the following Theorem.

Theorem 157 *If*

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{HPO}"] & B \arrow[d] \\ C \arrow[r] & D, \end{tikzcd}$$

is a homotopy pullback squares in which $A \to B$ is an n-equivalence and $A \to C$ is an m-equivalence, where (at least one of?) $n, m \geq 2$, then

$$\operatorname{conn}(A \to P) \geq n + m - 2.$$

The requirement that at least one of $n, m \geq 2$ is not much of a loss, since if it fails, the conclusion is only that the induced map is a 0-equivalence, which is automatically true for maps of path-connected spaces.

Problem 15.18

(a) Suppose f and g are weakly equivalent maps; show their homotopy fibers are weakly equivalent, and in particular, they have the same connectivity.

(b) Show that we can replace $A \to B$ with an inclusion of a CW complex into a larger CW complex built by attaching cells of dimension at least $n+1$.

Let A_k be the k^{th} step in the construction of B from A. Then D is likewise built in stages C_k from C, and we have a pointwise k-equivalence of squares

$$\begin{tikzcd}
A \arrow[rr, dotted] \arrow[dr] \arrow[dd] & & A_k \arrow[dr] \arrow[dd] & \\
& B \arrow[rr, dotted] \arrow[dd] & & B \arrow[dd] \\
C \arrow[rr, dotted] \arrow[dr] & & C_k \arrow[dr] & \\
& D \arrow[rr, dotted] & & D.
\end{tikzcd}$$

Problem 15.19

(a) Show that the squares

$$\begin{tikzcd}
A_k \arrow[r] \arrow[d] & B \arrow[d] \\
C_k \arrow[r] & D
\end{tikzcd}$$

are homotopy pushout squares.

(b) Show that if one square maps by a pointwise k-equivalence into another, then the induced map of iterated fibers is a $(k-2)$-equivalence.

Problem 15.20 Prove Theorem 157.

Exercise 15.21 To what extent can the simply-connected hypothesis in Theorem 157 be removed or relaxed?

15.5 Pairs and Triads

To Do:

1. Discuss notation and theorems; history.

2. Discuss generalization to cubes; put forward as challenge; discuss the tricky part.
3. Use to prove Barratt/Hopkins/Bousfield.

15.6 Projects

Section Subsets. (This can be a project) A fibration $p : E \to B$ identifies certain important subsets: those subsets $U \subseteq B$ over which which a section $s_U : U \to E$ exists. The Ganea construction has a very useful property in this regard.

PROBLEM 15.22 Show that if $U \subseteq B$, then the Ganea construction $q : E/F \to B$ has a homotopy section over U if and only if

$$U = V \cup C$$

where $p : E \to B$ has a section over V and the inclusion $i : C \hookrightarrow B$ is nullhomotopic.

Let $p_0 : G_0(X) \to X$ be the path fibration $\mathcal{P}(X) \to X$. Inductively define $p_{n+1} : G_{n+1}(X) \to X$ to be the result of applying the Ganea construction to $p_n : G_n(X) \to X$.

PROBLEM 15.23

(a) Show that $\mathrm{cat}(X) \leq n$ if and only if p_n has a section.

(b) Suppose X is $(n-1)$-connected and d-dimensional. How large can $\mathrm{cat}(X)$ possibly be?

Part IV

Algebraic Topology

Chapter 16

Some Computations in Homotopy Theory

Having now set up the fundamentals of Homotopy Theory, we now begin to apply it. In this chapter we do some explicit computations involving homotopy sets $[X, Y]$, and apply those calculations to prove topological theorems.

We begin by studying the degree of a reflection, and then of the antipodal map. We use this to prove results about fixed points of self-maps of spheres. We also determine the degree of the twisting homeomorphism $S^n \wedge S^m \to S^m \wedge S^n$, which appears ubiquitously in Algebraic Topology in the form of the Milnor Sign Convention.

We show that maps from one wedge of n-spheres to another are fully described by certain integer matrices, and use this fact to produce an example of a noncontractible CW complex X whose suspension is trivial. Then we study Moore spaces.

In the final two sections, we define the smash product pairing of homotopy groups, and use it to determine the smallest nontrivial homotopy group of a smash product. This is applied to define certain maps between Eilenberg-Mac Lane spaces that will be crucial in our development of cohomology algebras in the next chapter.

16.1 The Degree of a Reflection

The degree of a map $S^n \to S^n$ was first defined by L. E. J. Brouwer in 1910. A map $f : S^n \to S^n$ induces a map

$$\begin{tikzcd}
\pi_n(S^n) \arrow[r, "f_*"] \arrow[d, equal] & \pi_n(S^n) \arrow[d, equal] \\
\mathbb{Z} \cdot [\mathrm{id}_{S^n}] \arrow[r] & \mathbb{Z} \cdot [\mathrm{id}_{S^n}]
\end{tikzcd}$$

which is multiplication by some integer d, which we call the **degree** of f. On the face of it, this definition applies only to pointed maps, but it can be easily extended to unpointed maps.

prob:UnpointedDegree

Problem 16.1 Show that the forgetful map $[S^n, S^n] \to \langle S^n, S^n \rangle$ is bijective for $n \geq 1$, so degree makes sense for unpointed maps, too.

Degree behaves well with respect to standard operations.

Problem 16.2

(a) What is the degree of $f + g$?

(b) What is the degree of $f \circ g$?

Now we come to our first interesting computation: the degree of a reflection.

prob:ReflectionDegree

Problem 16.3

(a) Let $P_0, P_1 \subseteq \mathbb{R}^{n+1}$ be hyperplanes through the origin, let $R_0, R_1 : S^n \to S^n$ be the reflections through P_1 and P_2, respectively. Show that R_0 and R_1 are freely homotopic (i.e., they are homotopic as unpointed maps).

(b) What is the degree of a reflection?

(c) The **antipodal map** is the map $\alpha : S^n \to S^n$ given by $\alpha(x) = -x$. Express it in terms of reflections and determine its degree.

(d) For which n is $\alpha \simeq \mathrm{id}_{S^n}$?

Some Topological Applications. Our knowledge of the degree of a reflection, together with some simple topology, leads to some important conclusions about self-maps of spheres.

Problem 16.4

(a) Let $x, y \in S^n$ and assume that $x \neq -y$. Write down an explicit parametrization for the shortest arc on the sphere joining x to y. What happens when $x = -y$?

(b) Suppose $f : S^n \to S^n$ is a map so that $f(x) \neq -x$ for every $x \in S^n$. Show that $f \simeq \mathrm{id}_{S^n}$.

(c) Suppose $f : S^n \to S^n$ is a map with no fixed points (i.e. such that $f(x) \neq x$ for every x). Show that $f \simeq \alpha$.

Now you can derive some nice and surprising results. The first concerns fixed point free maps.

thm:abc

Theorem 158 (??) *If $f : S^{2n} \to S^{2n}$ has no fixed points, then there is some point $x \in S^{2n}$ such that $f(x) = -x$.*

PROBLEM 16.5

(a) Prove Theorem 158.

(b) Find examples of maps $S^{2n+1} \to S^{2n+1}$ without fixed points and with $f(x) \neq -x$ for all x.

Vector Fields on Spheres. A **vector field** on $S^n \subseteq \mathbb{R}^{n+1}$ is a function $v : S^n \to \mathbb{R}^{n+1}$ such that $v(x)$ is tangent to S^n at the point x.

EXERCISE 16.6 Show that a function $v : S^n \to \mathbb{R}^{n+1}$ is a vector field if and only if $v(x) \perp x$ for every $x \in S^n$.

thm:VectorFieldonSpheres

Theorem 159 *There is an everywhere nonzero vector field on $S^n \subseteq \mathbb{R}^{n+1}$ if and only if n is odd.*

PROBLEM 16.7

(a) If $n = 2k - 1$ is odd, then $S^n \subseteq \mathbb{R}^{n+1} \cong \mathbb{C}^k$. Use this to construct a nonzero vector field on S^n.

(b) Show that S^n has a nonzero vector field if and only if there is a function $f : S^n \to S^n$ such that $f(x) \perp x$ for all $x \in S^n$.

(c) Show that if S^n has a nonzero vector field, then n must be odd.

The Milnor Sign Convention. Let $T : S^n \wedge S^m \to S^m \wedge S^n$ be the **twist map** that interchanges smash factors. We identify $S^n \wedge S^m$ with S^{n+m} by iterating the homeomorphism $\Sigma S^n \cong S^{n+1}$ of Chapter **??**. Thus we obtain standard homeomorphisms

$$S^n \wedge S^m \cong (S^1)^{\wedge n} \wedge (S^1)^{\wedge m} \cong (S^1)^{\wedge(n+m)} \cong S^m \wedge S^n.$$

Thus we have a commutative square

$$\begin{tikzcd}
S^n \wedge S^m \arrow[r, "T"] \arrow[d, "\cong"'] & S^m \wedge S^n \arrow[d, "\cong"] \\
S^{n+m} \arrow[r, "\widetilde{T}"] & S^{n+m},
\end{tikzcd}$$

and we define the degree of T to be the degree of the corresponding map $\widetilde{T}$.

thm:MilnorSignConvention

Theorem 160 *The degree of* $T : S^n \wedge S^m \to S^m \wedge S^n$ *is* $(-1)^{nm}$.

The full statement of Theorem 160 follows easily from the case $n = m = 1$.

PROBLEM 16.8

(a) Think of S^1 as the quotient of I by identifying the endpoints. Then $S^1 \wedge S^1$ is a quotient of $I \times I$ by a certain equivalence relation. Describe the map T in terms of the square $I \times I$.

(b) Show that, under the identification of part (b), the map $T : S^2 \to S^2$ is reflection about a certain plane containing the x-axis.

(c) Prove Theorem 160 in the case $n = m = 1$, and derive the full statement.

Theorem 160 explains why algebraic operations derived from the smash product introduce signs. The general convention, called the **Milnor sign convention**, is that when two things x and y with dimensions $|x|$ and $|y|$ are moved past one another, a sign $(-1)^{|x|\cdot|y|}$ must be introduced, as we must do, for example when forming the tensor product of graded modules (see Appendix **??**). This commutativity formula is also called **graded commutativity**; when all the algebra in sight is graded (i.e., elements x have dimensions $|x|$ associated to them), the rule $xy = (-1)^{|x|\cdot|y|}yx$ is referred to simply as *commutativity.*

16.2 Maps Between Wedges of Spheres

In this section, we completely determine the set of all homotopy classes of maps $\bigvee_{\mathcal{I}} S^n \to \bigvee_{\mathcal{J}} S^n$ for $n \geq 2$ and explicitly describe their compositions. The case $n = 1$ is trickier, because the groups involved are nonabelian, and our conclusions are not as strong.

We will identify the maps between simply-connected wedges of n-spheres with certain groups of matrices. Write $M_{\mathcal{I}\times\mathcal{J}}(R)$ for the set of all matrices whose entries are in the ring R and are indexed on the product set $\mathcal{I} \times \mathcal{J}$.[1] For any $f : \bigvee_{i\in\mathcal{I}} S^n \to \bigvee_{j\in\mathcal{J}} S^n$, we can form the map $f_{ij} : S^n \to S^n$ by the

---

[1] This might give you pause when $\mathcal{I}$ or $\mathcal{J}$ (or both!) are infinite; but all such a matrix is is a function $A : \mathcal{I} \times \mathcal{J} \to R$.

diagram

$$\begin{tikzcd}
S^n \arrow[r, "f_{ij}"] \arrow[d, "\mathrm{in}_j"'] & S^n \\
\bigvee_{j\in\mathcal{J}} S^n \arrow[r, "f"] & \bigvee_{i\in\mathcal{I}} S^n \arrow[u, "q_i"']
\end{tikzcd}$$

where q_i is the map which is trivial on all spheres except the i^{th} one, and it is the identity on that sphere. The map f_{ij} represents an element of $\pi_n(S^n)$, which has a degree. We define a matrix $A = A(f) : \mathcal{I} \times \mathcal{J} \to \mathbb{Z}$ by setting $A(i, j) = \deg(f_{ij})$.

Problem 16.9 Show that the matrix $A(f) \in M_{\mathcal{I}\times\mathcal{J}}(\mathbb{Z})$ has only finitely many nonzero entries in each row.

We write $\overline{M}_{\mathcal{I}\times\mathcal{J}}(R)$ for the (additive) subgroup of $M_{\mathcal{I}\times\mathcal{J}}(R)$ consisting of those matrices with only finitely many nonzero entries in each row.

prop:WedgeSphereMaps

Proposition 161 *The function*

$$A : \left[\bigvee_{\mathcal{J}} S^n, \bigvee_{\mathcal{I}} S^n\right] \to \overline{M}_{\mathcal{I}\times\mathcal{J}}(\mathbb{Z})$$

is an isomorphism of (additive) abelian groups.

Problem 16.10

(a) Show that the inclusion in : $\bigvee_{\mathcal{I}} S^n \to \prod_{\mathcal{I}} S^n$ of the wedge into the **weak product**[2] is a $(2n-1)$-equivalence.

(b) Prove Proposition 161.

Problem 16.11

(a) Determine $\pi_n(\bigvee_{\mathcal{I}} S^n)$.

(b) Show that the map

$$\left[\bigvee_{\mathcal{J}} S^n, \bigvee_{\mathcal{I}} S^n\right] \to \mathrm{Hom}\left(\pi_n\left(\bigvee_{\mathcal{J}} S^n\right), \pi_n\left(\bigvee_{\mathcal{I}} S^n\right)\right)$$

is an isomorphism, so that $f \simeq g$ if and only if $f_* = g_*$

Problem 16.12 If $\mathcal{I} = \mathcal{J}$, then we can compose elements of $[\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n]$ with each other. Show that this makes the additive group $[\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n]$ into a ring, and verify that $\alpha : [\bigvee_{\mathcal{I}} S^n, \bigvee_{\mathcal{I}} S^n] \to \overline{M}_{\mathcal{I},\mathcal{I}}(\mathbb{Z})$ is a ring isomorphism (including verifying that $\overline{M}_{\mathcal{I},\mathcal{I}}(\mathbb{Z})$ is a ring).

---

[2] the colimit of the finite products

The Non-Simply-Connected Case. Now we want to study maps $f : \bigvee_{\mathcal{J}} S^1 \to \bigvee_{\mathcal{I}} S^1$. Since the domain of f is a sum, f is determined by the maps $f_j = f \circ \mathrm{in}_j : S^1 \to \bigvee_{\mathcal{I}} S^1$. But now, in contrast with the simply-connected case, the target cannot be replaced with a product, so our analysis stops here: f is determined by the $\mathcal{J}$-tuple

$$(f \circ \mathrm{in}_1, f \circ \mathrm{in}_2, \ldots) \in \prod_{\mathcal{J}} \pi_1\left(\bigvee_{\mathcal{I}} S^1\right).$$

In this formula, each map $f \circ \mathrm{in}_j$ is an element of $\pi_1(\bigvee_{\mathcal{I}} S^1)$. Since this is a free group on generators $\{\mathrm{in}_i \mid i \in \mathcal{I}\}$, the element $f \circ \mathrm{in}_j$ has a unique expression as a reduced word in the symbols $\{\mathrm{in}_i, \mathrm{in}_i^{-1} \mid i \in \mathcal{I}\}$. This is about as far as we can push the argument in this case.

If we suspend f, the factors in our word commute, so we can collect terms and add the exponents.

need picture!

Problem 16.13 The Klein bottle K is the space obtained from the square $I \times I$ by gluing the edges as indicated in the picture.

(a) Show that K is the cofiber of a map $S^1 \to S^1 \vee S^1$. Describe this map explicitly as an element of the free group $\pi_1(S^1 \vee S^1) = F(a, b)$.

(b) Give a presentation for $\pi_1(K)$. Is this group abelian?

(c) Determine the homotopy type of ΣK. What is $\pi_2(\Sigma K)$?

This approach can be used to to show that there are indeed monsters. We take for granted that the group

$$G = \langle a, b, c, d \mid bab^{-1}a^{-2}, cbc^{-1}b^{-2}, dcd^{-1}c^{-2}, ada^{-1}d^{-2}\rangle$$

is nontrivial [???].[3]

prob:TrivialSuspension

Problem 16.14 Write a, b, c and d for the inclusions $\mathrm{in}_1, \mathrm{in}_2, \mathrm{in}_3$ and $\mathrm{in}_4 \in \pi_1(\bigvee_1^4 S^1)$, and set

$$r_1 = bab^{-1}a^{-2}, \quad r_2 = cbc^{-1}b^{-2}, \quad r_3 = dcd^{-1}c^{-2} \qquad \text{and} \qquad r_4 = ada^{-1}d^{-2}.$$

Let $f : \bigvee_1^4 S^1 \to \bigvee_1^4 S^1$ be the map given by the 4-tuple (r_1, r_2, r_3, r_4), and let X be its cofiber.

---

[3] From Bob Bruner: MR0038348 (12,390c) 20.0X Higman, Graham. *A finitely generated infinite simple group.* J. London Math. Soc. 26, (1951). 61–64.

Suppose that the group G is generated by elements a, b, c, d subject to the relations:

$$a^b = a^2, \quad b^c = b^2, \quad c^d = c^2 \qquad \text{and} \qquad d^a = d^2$$

This group G is shown to be infinite and to be without normal subgroups of finite index, except G. As a finitely generated group, G possesses maximal normal subgroups $N \neq G$; and it is clear that G/N is an infinite simple group.

(a) Show that $X \not\simeq *$.

(b) Show that $\Sigma X \simeq *$.

PROBLEM 16.15 Show that for each n, there is a path-connected CW complex Y_n such that Y_n is not simply-connected but ΣY_n is n-connected but not $(n+1)$-connected.

PROBLEM 16.16 Show that if $\Sigma X \simeq *$, then for any path-connected CW complex Y, $X \wedge Y \simeq *$.

PROBLEM 16.17 Give an example of a homotopy pushout square

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in which $B \to D$ is a homotopy equivalence and $A \to C$ is not a homotopy equivalence.

16.3 Moore Spaces

We have discussed Moore spaces for cyclic groups in Section **??**. In this section we develop a more general theory, in which the Moore spaces occupy a position that is (nearly) dual to that of Eilenberg-Mac Lane spaces.

Definition of Moore Spaces. Let G be an abelian group with free resolution

$$0 \longrightarrow F_1 \xrightarrow{d} F_0 \longrightarrow G \longrightarrow 0\ ,$$

where $F_0 = \bigoplus_{\mathcal{I}} \mathbb{Z}$ and $F_1 = \bigoplus_{\mathcal{J}} \mathbb{Z}$. From our study of maps between wedges of spheres, we see that there is a unique homotopy class of maps $\delta : \bigvee_{\mathcal{J}} S^n \to \bigvee_{\mathcal{I}} S^n$ whose induced on π_n is $d : F_1 \to F_0$. We study its cofiber $M_n(G)$.

PROBLEM 16.18 Determine $\pi_k(M_n(G))$ for $k \leq n$.

prob:StableMooreGroups

PROBLEM 16.19 Show that for $k \leq 2n-2$, there are exact sequences

$$0 \longrightarrow G \otimes \pi_k(S^n) \longrightarrow \pi_k(M_n(G)) \longrightarrow \mathrm{Tor}(G, \pi_{k-1}(S^n)) \longrightarrow 0.$$

In what sense are they natural?

thm:MooreSpacesDefined

Theorem 162 *The space $M_n(G)$ depends only on the number n and the group G, and not on the choice of free resolution of G.*

To prove Theorem 162, suppose $0 \to H_1 \to H_0 \to G \to 0$ is another free resolution of the group G. Realize the map $H_1 \to H_0$ as a map $\epsilon : V_1 \to V_0$ of wedges of spheres, and let N be its cofiber.

Problem 16.20

(a) Show that there is a commutative square

$$\begin{tikzcd} W_1 \arrow[r] \arrow[d] & W_0 \arrow[d] \\ V_1 \arrow[r] & V_0. \end{tikzcd}$$

(b) Prove Theorem 162 by realizing the square of part (a) using maps of wedges of spheres.

Since the homotopy type of the cofiber only depends on the dimension n and the group G, we denote it $M_n(G)$. This spaces is called the **Moore space** of type (G, n).

Problem 16.21 Show that $M_n(G)$ is a suspension (where $n \geq 2$).

Exercise 16.22 Discuss the issues involved in defining $M_1(G)$.

prop:MooreSkeleta

Proposition 163 *If X is $(n-1)$-connected with $\pi_n(X) = G$, then*

(a) The map $\phi : [M_n(H), X] \to \mathrm{Hom}(H, G)$ given by $\phi : f \mapsto \pi_n(f)$ is surjective, and

(b) X has an $(n+1)$-skeleton of the form $X_{n+1} = M_n(G) \vee \bigvee S^{n+1}$.

Problem 16.23 Prove Proposition 163.

Exercise 16.24 What do you need to know about H to know that ϕ is bijective? Why is this result different from the corresponding one (Proposition **??** for Eilenberg-Mac Lane spaces?

The next problem generalizes the result of Section **??**.

prop:MooreSpaceUCT

Proposition 164 *Show that there are natural exact sequences*

$$0 \longrightarrow \pi_{n+1}(X) \otimes G \longrightarrow [M_n(G), X] \longrightarrow \mathrm{Tor}(\pi_n(X), G) \longrightarrow 0.$$

Problem 16.25 Prove Proposition 164 by applying the functor $[\,?\,, X]$ to a cofiber sequence that defines $M_n(G)$.

The groups $[M_n(G), X]$ are sometimes called the homotopy groups of X with **coefficients** in G, and denoted

$$\pi_{n+1}(X; G) = [M_n(G), X].$$

Since, for fixed X, the groups in this sequence are contravariant functors of G, Hilton used the notation $\pi_{n+1}(G; X)$ for these groups, and called G the **contraficients**; this didn't really catch on. The index shifting here has led a number of authors – including the eponymous Moore – to use the notation $P^{n+1}(G)$ for what we call $M_n(G)$; with this notation, $\pi_n(X; G) = [P^n(G), X]$.

Contractible Smash Products. In the previous section, we built an example of a noncontractible space X with $S^1 \wedge X \simeq *$. The construction depended on strange nonabelian features of the fundamental group $\pi_1(X)$. Since things seem to behave much more nicely with simply-connected spaces (the homotopy groups are abelian, for instance), it might seem reasonable to guess that if X and Y are simply-connected and not contractible then $X \wedge Y$ is not contractible either. We finish this section by showing that simply connected spaces *can* have a contractible smash product.

PROBLEM 16.26 Let $N = M(\mathbb{Z}/p, n)$ and $M = M(\mathbb{Z}/q, m)$, where p and q are two distinct prime numbers.

(a) Since M is a suspension, $[M, M]$ has a natural group structure. Show that $(p \cdot \mathrm{id}_{S^n}) \wedge \mathrm{id}_M$ is homotopic to $p \cdot \mathrm{id}_{\Sigma^n M}$.

(b) Show that if $k \leq 2(n + m) - 1$, then the map $\pi_k(\Sigma^n M) \to \pi_k(\Sigma^n M)$ induced by $p \cdot \mathrm{id}_{\Sigma^n M}$ is given by multiplication by p. How much of an equivalence is it?

(c) Show that $N \wedge M \simeq *$, so that in which

$$\mathrm{conn}(N \wedge M) > \mathrm{conn}(N) + \mathrm{conn}(M) + 1.$$

(d) Give an example of a sequence $A \to B \to C$ which is simultaneously a fiber sequence and a cofiber sequence.

Later we will see that, for simply-connected spaces, this kind of algebraic incompatibility is the *only* way to have a contractible smash product.

16.4 Homotopy Groups of a Smash Product

Let X and Y be any two spaces. We can take any two elements $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(Y)$ (so $\alpha : S^n \to X$ and $\beta : S^m \to Y$) and smash them together to give us a map

$$\alpha \wedge \beta : S^n \wedge S^m \to X \wedge Y.$$

Using our standard homeomorphism $S^n \wedge S^m \cong S^{n+m}$, we can view $\alpha \wedge \beta$ as an element of $\pi_{n+m}(X \wedge Y)$. We have defined a function (map of *pointed* *sets*)

$$\pi_n(X) \times \pi_m(Y) \to \pi_{n+m}(X \wedge Y),$$

which we call the **smash product pairing** of homotopy groups. It is called an **external product** because the product of two elements does not land back inside the same collection of homotopy groups that they came from.

Problem 16.27 Check that the homotopy class of $\alpha \wedge \beta$ only depends on the homotopy classes of α and β, and that the smash product is natural with respect to both variables.

Algebraic Properties of the Smash Product. The smash product operation is bilinear, associative and (graded) commutative. Proving bilinearity boils down to the distributivity law for smashes of maps over wedges of maps.

prob:dist

Problem 16.28 Let $\alpha : A \to X$, $\beta : B \to Y$ and $\gamma : C \to Z$. Then the diagram

$$\begin{tikzcd}
A \wedge (B \vee C) \arrow[r, "\alpha\wedge(\beta\vee\gamma)"] \arrow[d, "\cong"'] & X \wedge (Y \vee Z) \arrow[d, "\cong"] \\
(A \wedge B) \vee (A \wedge C) \arrow[r, "(\alpha\wedge\beta)\vee(\alpha\wedge\gamma)"] & (X \wedge Y) \vee (X \wedge Z)
\end{tikzcd}$$

is commutative.

prop:SmashBilinear

Proposition 165 *The smash product defines a natural group homomorphism*

$$\wedge : \pi_n(X) \otimes \pi_m(Y) \to \pi_{n+m}(X \wedge Y).$$

Let F be the free abelian group generated by the symbols $\alpha \otimes \beta$ with $\alpha \in \pi_n(X)$ and $\beta \in \pi_m(Y)$. Let R be the subgroup generated by the relations so that $F/R = \pi_n(X) \otimes \pi_m(Y)$. We can define $\wedge : F \to \pi_{n+m}(X \wedge Y)$ by the rule $\wedge(\alpha \otimes \beta) = \alpha \wedge \beta$. To show that it induces a homomorphism from the tensor product, we just have to show that $R \subseteq \ker(\wedge)$, or equivalently, that $\wedge$ is linear in each coordinate.

Problem 16.29 Verify Proposition 165.

prop:GradedComm

The smash product operation is also commutative, in the graded sense.

Proposition 166 *For any X and Y, the diagram*

$$\begin{tikzcd}
\pi_n(X) \otimes \pi_m(Y) \arrow[r, "\wedge"] \arrow[d, "(-1)^{nm}\tau"'] & \pi_{n+m}(X \wedge Y) \arrow[d, "T_*"] \\
\pi_m(Y) \otimes \pi_n(X) \arrow[r, "\wedge"] & \pi_{n+m}(Y \wedge X)
\end{tikzcd}$$

commutes.

PROBLEM 16.30 Prove Proposition 166.

Nondegeneracy. So far, nothing we have said guarantees that this smash product pairing is nontrivial – it might be that $\alpha \wedge \beta \simeq *$ for every α and β as far as we know! But this is actually far from true — indeed, the tensor product models the smash product in low dimensions.

prop:PiSmash

Proposition 167 *Suppose* X *is* $(n-1)$*-connected and* Y *is* $(m-1)$*-connected, where* $m, n \geq 1$ *and* $\pi_n(X)$ *and* $\pi_m(Y)$ *are abelian groups. Then the smash product map*

$$\wedge : \pi_n(X) \otimes \pi_m(Y) \to \pi_{n+m}(X \wedge Y).$$

is a natural isomorphism.

PROBLEM 16.31

(a) Show that if $X = S^1$ then $\wedge : \pi_1(X) \otimes \pi_m(Y) \to \pi_{m+1}(S^1 \wedge Y)$ is simply the suspension homomorphism, and verify the proposition in this case.

(b) Verify the proposition for $X = S^n$.

(c) Verify the proposition for $X = M_n(G)$.

HINT Use the cofiber sequence that defines $M_n(G)$.

(d) Prove Proposition 167.

16.5 Smash Products of Eilenberg-Mac Lane Spaces

We use our study of smash products to build some maps involving Eilenberg-Mac Lane spaces. Proposition 167 gives an isomorphism

$$\pi_{n+m}(K(G,n) \wedge K(H,m)) \cong G \otimes H$$

which we use to relate Eilenberg-Mac Lane spaces for G, H and $G \otimes H$.

Let $n, m \geq 1$ and let G and H be abelian groups. According to Exercise **??**, $K(G,n) \wedge K(H,m)$ is $(n+m-1)$-connected, and according to Proposition 167, $\pi_{n+m}(K(G,n) \wedge K(H,m)) \cong G \otimes H$. Proposition 140, ensures that there is a unique map

$$c : K(G,n) \wedge K(H,m) \to K(G \otimes H, n+m)$$

which induces the composite isomorphism

$$\pi_{n+m}(K(G,n) \wedge K(H,m)) \xrightarrow{\cong} G \otimes H \xrightarrow{\cong} \pi_{n+m}(K(G \otimes H, n+m))$$

given by combining Proposition 167 with the isomorphisms $\pi_n(K(G,n)) \cong G$, $\pi_m(K(H,m)) \cong H$ and $\pi_{n+m}(K(G \otimes H, n+m) \cong G \otimes H$ that come as part of the structure of Eilenberg-Mac Lane spaces. Now let

$$T : K(G,n) \wedge K(H,m) \to K(H,m) \wedge K(G,n)$$

be the twist map, and consider the diagram

$$\begin{tikzcd}
K(G,n) \wedge K(H,m) \arrow[r] \arrow[d, "T"'] & K(G \otimes H, n+m) \arrow[d, dotted, "t"] \\
K(H,m) \wedge K(G,n) \arrow[r] & K(H \otimes G, n+m).
\end{tikzcd}$$

Since $K(G,n) \wedge K(H,m)$ contains the $(n+m+1)$-skeleton of $K(G \otimes H, n+m)$, we can apply Proposition 137 and conclude that there is a unique map $t : K(G \otimes H, n+m) \to K(G \otimes H, n+m)$ making the diagram commute.

PROBLEM 16.32 Study the associativity of the maps c.

prob:EMTwist

PROBLEM 16.33 According to Proposition 140, the map t corresponds to a homomorphism $\phi : G \otimes H \to H \otimes G$. Show that $\phi = (-1)^{nm}\tau$, where τ is the unsigned twist map $\tau(\alpha \otimes \beta) = \beta \otimes \alpha$.

Disconnected Eilenberg-Mac Lane Spaces. First of all, if G is an abelian group, then we define $K(G,0)$ to be the set G, with the discrete topology. Then $K(G,0)$ is automatically a CW complex, and it is automatically a group object in $\mathrm{H}\mathcal{T}_*$, because it is a group!

Since $K(G,0)$ is just a big wedge of copies of S^0, one for each nonidentity element of G, when we form the smash product $K(G,0) \wedge K(H,n)$ we get

$$K(G,0) \wedge K(H,n) = \left(\bigvee_{g \neq 1 \in G} S^0 \right) \wedge K(H,m)) = \bigvee_{g \neq 1 \in G} K(H,m).$$

Applying π_m to this space yields $\bigoplus H$, and not $G \otimes H$.

Nevertheless, we would like to have a map

$$c : K(G,0) \wedge K(H,m) \to K(G \otimes H, m).$$

Since we have $K(G,0) \wedge K(H,m)$ written as a wedge, we just need to define c on each wedge summand – in other words, we need to define maps $K(H,m) \to K(G \otimes H, m)$, one for each nonidentity element $g \in G$. But the answer is right in front of us! If $g \in G$, then we can define

$$\phi_g : H \to G \otimes H \quad \text{by the formula} \quad \pi_g(h) = g \otimes h.$$

Since we know that

$$[K(G,n), K(G\otimes H,n)] \cong \hom(G, G\otimes H),$$

there is a unique homotopy class $c_g \in [K(G,n), K(G\otimes H,n)]$ such that $(c_g)_* = \phi_g$. We define

$$c = (c_g \,|\, g \in G - \{1\}) : K(G,0)\wedge K(H,m) \to K(G\otimes H,m).$$

This is the map we want. I'll leave it to you to check that the same construction works equally well to define a nice map $K(G,n)\wedge K(H,0) \to K(G\otimes H,n)$.

Ring Structures and Eilenberg-Mac Lane Spaces. Now let R be a ring. Since the multiplication $R\times R \to R$ is bilinear, it defines a new map from the tensor product, $R\otimes R \to R$, which corresponds to a homotopy class

$$m : K(R\otimes R,n) \to K(R,n).$$

We need to understand the composite map (call it μ)

$$K(R,0)\wedge K(R,n) \xrightarrow{c} K(R\otimes R,n) \xrightarrow{m} K(R,n).$$

Since $K(R,0)\wedge K(R,n) = \bigvee_{r\neq 0\in R} K(R,n)$, we just need to determine the restriction of μ to the r^{th} summand.

prob:RingEML

PROBLEM 16.34 If $r \in R$, then multiplication by r defines a homomorphism $\phi_r : R \to R$, and hence a map $f_r : K(R,n) \to K(R,n)$. Show that the composite

$$K(R,n) \xrightarrow{\text{in}_r} K(R,0)\wedge K(R,n) \xrightarrow{c} K(R\otimes R,n) \xrightarrow{m} K(R,n)$$

is homotopic to f_r.

Finally, we mention that if R has a multiplicative identity element, then $\phi_1 = \text{id}$ and so $f_r \simeq \text{id}$, which means that the diagram

$$\begin{tikzcd}
K(R,n) \arrow[r, "\text{in}_1"] \arrow[dr, "\text{id}"'] & K(R,0)\wedge K(R,n) \arrow[d, "m\circ c"] \\
& K(R,n)
\end{tikzcd}$$

is homotopy commutative.

Chapter 17

Cohomology

In this chapter we introduce an extremely powerful approach to the study of homotopy theory, namely cohomology theories. A cohomology theory is a collection of functors that are related to one another via the suspension operation. The first, and most important, example is called **ordinary cohomology**, which is defined in terms of Eilenberg-Mac Lane spaces.

We construct a multiplicative structure involving cohomology theories which, when we use coefficients in a ring R, yields an R-algebra structure on the cohomology of spaces. We related cohomology with various coefficients and prove a simple result about the cohomology of a product (smash or cartesian) of two spaces.

17.1 Cohomology

Eilenberg-MacLane spaces are good spaces to map **into**, in contrast to spheres, Moore spaces, etc. This suggests that the sets $[X, K(G,n)]$ deserve a special notation, and they do.

defn:OrdinaryCohomology

Definition 168 Let G be an abelian group. The n^{th} **cohomology of X with coefficients in G** is the abelian group

$$\widetilde{H}^n(X;G) = [X, K(G,n)].$$

It is convenient to define $\widetilde{H}^n(X;G) = 0$ for all $n < 0$.

We use the notation $\widetilde{H}^*(X;G)$ to denote the whole collection of cohomology groups. An element $u \in \widetilde{H}^n(X;G)$ is said to be an **n-dimensional cohomology class**. The dimension of $u \in \widetilde{H}^*(X;G)$ is denoted $|u|$.

Intuitively, $\widetilde{H}^n(X;G)$ measures the n-dimensional features of X. This suggests, in particular, that $\widetilde{H}^0(X;G)$ depends only on the path components of X.

Exercise 17.1 Show that $\widetilde{H}^0(X;G) \cong G \times G \times \cdots \times G$, where the number of factors is one less than the number of path components of X.

The fundamental properties of cohomology were laid down by Eilenberg and Steenrod in their book [`?Eilenberg-Steenrod`]. These are known as the **Eilenberg-Steenrod axioms**. Let us say that a functor $F : \mathcal{T} \to \mathcal{C}$ is a **homotopy functor** if whenever $f \simeq g$, $F(f) = F(g)$.

Exercise 17.2 Show that if F is a homotopy functor, and $X \simeq Y$, then $F(X) \cong F(Y)$.

Here are the main properties of cohomology.

thm:cohoprop

Theorem 169 *For any abelian group G,*

(a) $\widetilde{H}^n(\,?\,;G)$ is a contravariant homotopy functor $\mathcal{T}_* \to$ AB$\mathcal{G}$.

(b) There is a natural isomorphism $\widetilde{H}^n(\,?\,;G) \cong \widetilde{H}^{n+1}(\Sigma\,?\,;G)$.

(c) Let $A \to B \to C$ be a cofiber sequence. Then there is a natural long exact[1] sequence

$$\cdots \to \widetilde{H}^n(C) \to \widetilde{H}^n(B) \to \widetilde{H}^n(A) \to \widetilde{H}^{n+1}(C) \to \cdots .$$

(Here we're beginning to use the usual convention: don't write down the group G unless you need to; also, the maps are the only reasonable ones, so we omit their labels).

The list of properties given by Eilenberg and Steenrod is significantly longer, but this is because the language of categories and functors was new at the time of their formulation, so there were axioms explicitly laying out the functoriality and naturality.

Problem 17.3 Prove Theorem 169, being especially careful to explain what it means for the long exact sequence of part (c) to be natural.

Exercise 17.4 Show that parts Theorem 169(c) implies Theorem 169(b).

Any sequence of functors $\{\widetilde{h}^n\}$ satisfying the properties of Theorem 169 is called a **cohomology theory**. The graded abelian group $\widetilde{h}^*(S^0)$ is know as the **coefficients** of the cohomology theory $\widetilde{h}^*$. If this graded group is concentrated only in degree 0 (i.e. $\widetilde{h}^k(S^0) = 0$ for $k \neq 0$), then $\widetilde{h}^*$ is known

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[1] There is no need to worry about the $\pm f$ that appear in the long cofiber sequence because $\ker(-f^*) = \ker(f^*)$ and $\mathrm{Im}(-f^*) = \mathrm{Im}(f^*)$.

as an **ordinary cohomology theory**. We will show later that ordinary cohomology theories are essentially determined by their **coefficient group** $\widetilde{h}^0(S^0)$.

Cohomology theories are well suited to studying domain-type spaces. In particular, the cohomology of a homotopy pushout is comparatively easy to compute.

Problem 17.5 (Mayer-Vietoris Sequence) Let

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & B \arrow[d, "k"] \\ C \arrow[r, "l"] & D \end{tikzcd}$$

be a homotopy pushout square, and let $\widetilde{h}^*$ be a cohomology theory. Show that there is a natural long exact sequence

$$\cdots \longrightarrow \widetilde{h}^n(D) \xrightarrow{(i^*,j^*)} \widetilde{h}^n(B) \oplus \widetilde{h}^n(C) \xrightarrow{(l^*,k^*)} \widetilde{h}^n(A) \longrightarrow \widetilde{h}^{n+1}(D) \longrightarrow \cdots.$$

Problem 17.6 Let's do some computation.

(a) Show that
$$\widetilde{H}^n(S^k;G) = \begin{cases} G & \text{if } n = k \\ 0 & \text{otherwise.} \end{cases}$$

(b) Consider the space $\mathbb{C}\mathrm{P}^2 = S^2 \cup D^4$ and show
$$\widetilde{H}^n(\mathbb{C}\mathrm{P}^2;G) = \begin{cases} G & \text{if } n = 2 \text{ or } 4 \\ 0 & \text{otherwise.} \end{cases}$$

Now we study the relationship between the cells of a CW complex and its cohomology groups.

prob:CohoCells

Problem 17.7

(a) Show that the map $X_{n+1} \to X$ induces an isomorphism
$$\widetilde{H}^n(X;G) \to \widetilde{H}^n(X_{n+1};G).$$

(b) Let X be a CW complex with no cells of dimension m. Show that $\widetilde{H}^m(X;G) = 0$.

Hint It is enough to check it for X_{m+1}. Work by induction on the skeleta of X, using the long exact sequence of the cofiber sequence $\bigvee S^n \to X_n \to X_{n+1}$.

(c) Assume that X is an $(n-1)$-connected CW complex. Show that $\widetilde{H}^k(X;G) = 0$ for $k < n$.

(d) Suppose $\dim(X) < n$. Show that $\widetilde{H}^n(X;G) = 0$.

prob:CohoCells

Problem 17.8 Let X be a CW complex.

(a) Show that the map $X \to X/X_{n-2}$ induces an isomorphism

$$\widetilde{H}^n(X/X_{n-2};G) \to \widetilde{H}^n(X;G).$$

(b) Show that $\widetilde{H}^n(X;G) \cong \widetilde{H}^n(X_{n+1}/X_{n-2};G)$.

(c) Assume that X is a CW complex with cells only in dimensions that are divisible by 4. Describe the cohomology groups $\widetilde{H}^m(X;\mathbb{Z})$. Can there be any elements of finite order in these cohomology groups?

Working backwards, we can derive some topological information about a space from knowledge of its cohomology.

Problem 17.9

(a) Show that any CW decomposition of $\mathbb{C}\mathrm{P}^2$ must be at least 4-dimensional.

(b) Compute the cohomology of the space constructed in Problem 16.14.

(c) Let $n \geq 2$, and determine $H^*(\Omega S^{n+1};G)$. What can you say about the dimension of ΩS^{n+1}?

Problem 17.10 Let p be a prime number.

(a) Compute $\widetilde{H}^*(M_n(\mathbb{Z}/p);\mathbb{Z})$.

(b) Now compute $\widetilde{H}^*(M_n(\mathbb{Z}/p);\mathbb{Z}/p)$.

Cohomology for Unpointed Spaces. For cohomology of unpointed spaces, we define

$$H^n(X;G) = \widetilde{H}^n(X^+;G) \cong \langle X, K(G,n)\rangle.$$

The cohomology theory $\widetilde{H}^n$ is called **reduced cohomology**, and H^n is called the **unreduced cohomology**.

Problem 17.11 Show that if X is a pointed space and $n \geq 1$, then

$$\widetilde{H}^n(X;G) \cong H^n(X;G) = \widetilde{H}^n(X_+;G).$$

Detecting Connectivity with Cohomology. The J. H. C. Whitehead Theorem tells us that a map $f : X \to Y$ between two CW complexes is an homotopy equivalence if and only if the induced maps $f_* : \pi_k(X) \to \pi_k(Y)$ are isomorphisms for all k. For simply-connected spaces, we can make a similar conclusion by studying the induced maps on cohomology.

:CohoConn

Proposition 170 *If X is simply-connected, then the following are equivalent*

1. *X is $(n-1)$-connected and not n-connected*
2. *$\widetilde{H}^k(X;G) = 0$ for all $k < n$ and all abelian groups G, and $\widetilde{H}^n(X;G) \neq 0$ for some group G.*

PROBLEM 17.12 Prove Proposition 170.

We can determine the connectivity of a map by applying Proposition 170 to its cofiber.

thm:CohoJHCW

Theorem 171 *Let X and Y be simply-connected CW complexes, and let $f : X \to Y$. The following are equivalent*

1. *If $f^* : \widetilde{H}^k(Y;G) \to \widetilde{H}^k(X;G)$ is an isomorphism for every value of $k < n$ and is injective for $k = n$, no matter what coefficient group G is used*
2. *f is an n-equivalence.*

PROBLEM 17.13 Prove Theorem 171.

HINT What can you say about C_f?

The Wedge Axiom. Let us consider the cohomology of a wedge $\bigvee_{\mathcal{J}} X_j$. The inclusions $\mathrm{in}_j : X_j \hookrightarrow \bigvee_{\mathcal{J}} X_j$ induce maps $\widetilde{h}^n(\bigvee_{\mathcal{J}} X_j) \to \widetilde{h}^n(X_j)$ and hence a canonical comparison map

$$w : \widetilde{h}^* \left(\bigvee_{\mathcal{J}} X_j\right) \xrightarrow{\cong} \prod_{\mathcal{J}} \widetilde{h}^*(X_j).$$

EXERCISE 17.14 Explain how to interpret w as a natural transformation.

PROBLEM 17.15

(a) Show that if $\mathcal{J}$ is finite, the map w is an isomorphism, no matter what cohomology theory $\widetilde{h}^*$ is used.

(b) Show that in our collection of examples ($\widetilde{H}^n(X;G) = [X, K(G,n)]$), the map w is an isomorphism for *all* wedges, finite or infinite.

A cohomology theory $\widetilde{h}^*$ satisfies the **wedge axiom** if for any (finite or infinite) wedge $\bigvee_{\mathcal{J}} X_j$, the canonical comparison map w is an isomorphism. There are cohomology theories that do not satisfy the Wedge Axiom (for infinite wedges).

Consider the telescope diagram

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{f_{n+1}} \cdots .$$

The **shift map** for this diagram is the map

$$\text{SHIFT} : \bigvee X_n \to \bigvee X_n$$

whose restriction to X_n is the composite $X_n \xrightarrow{f_n} X_{n+1} \xhookrightarrow[n+1]{\text{in}} \bigvee X_n$.

prob:WedgePushout

Problem 17.16 Let X be the homotopy colimit of the telescope diagram

$$X_0 \to X_1 \to \cdots \to X_n \to X_{n+1} \to \cdots .$$

Show that X is the homotopy pushout in the square

$$\begin{tikzcd}
(\bigvee X_n) \vee (\bigvee X_n) \arrow[r, "(\mathrm{id},\mathrm{SHIFT})"] \arrow[d, "\mathrm{FOLD}"'] \arrow[dr, phantom, "\mathrm{HPO}"] & \bigvee X_n \arrow[d] \\
\bigvee X_n \arrow[r] & X.
\end{tikzcd}$$

Hint This is a homotopy version of Problem 2.34.

Problem 17.17 Now consider the map of telescopes

$$\begin{tikzcd}
X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X_n \arrow[r] \arrow[d] & X_{n+1} \arrow[r] \arrow[d] & \cdots \\
Y_0 \arrow[r] & Y_1 \arrow[r] & \cdots \arrow[r] & Y_n \arrow[r] & Y_{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

with induced map $f : X \to Y$ of homotopy colimits. Let $\widetilde{h}^*$ be a cohomology theory that satisfies the Wedge Axiom, and assume that the induced maps $\widetilde{h}^*(Y_n) \to \widetilde{h}^*(X_n)$ are isomorphisms for all n. Show that $f^* : \widetilde{h}^*(Y) \to \widetilde{h}^*(X)$ is also an isomorphism.

17.2 Transformations of Cohomology Theories

A natural transformation $T : \widetilde{h}^* \to \widetilde{k}^*$ of cohomology theories is a sequence of natural transformations $T_n : \widetilde{h}^n \to \widetilde{k}^n$ such that for any cofiber sequence $A \to B \to C$, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \widetilde{h}^n(B) \arrow[r] \arrow[d] & \widetilde{h}^n(A) \arrow[r] \arrow[d, "T_n"] & \widetilde{h}^{n+1}(C) \arrow[r] \arrow[d, "T_{n+1}"] & \widetilde{h}^{n+1}(B) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \widetilde{k}^n(B) \arrow[r] & \widetilde{k}^n(A) \arrow[r] & \widetilde{k}^{n+1}(C) \arrow[r] & \widetilde{h}^{n+1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

commutes. The commutativity of most of this diagram is handled by the fact that each $T : \widetilde{h}^n \to \widetilde{k}^n$ is a natural transformation in the ordinary sense; but for the center square to commute, T must be compatible with the natural transformation

$$\widetilde{h}^n(\textsc{domain}(\,?\,)) \to \widetilde{h}^{n+1}(\textsc{cofiber}(\,?\,))$$

(of functors $\mathrm{map}(\mathcal{T}_*) \to \textsc{ab}\,\mathcal{G}$) that is implicit in the statement that the long exact sequence of a cofiber sequence is functorial.

prob:CohoIsoSphere

Problem 17.18 Let $T : \widetilde{h}^* \to \widetilde{k}^*$ be a natural transformation of cohomology theories. Show that the following are equivalent:

(a) $T : \widetilde{h}^*(S^n) \to \widetilde{k}^*(S^n)$ is an isomorphism for some n.

(b) $T : \widetilde{h}^*(S^n) \to \widetilde{k}^*(S^n)$ is an isomorphism for *all* n.

(c) T is an isomorphism for all finite CW complexes.

We can improve the conclusion of Problem 17.18 if our cohomology theories satisfy the Wedge Axiom.

Problem 17.19 Suppose $\widetilde{h}^*$ and $\widetilde{k}^*$ are cohomology theories that satisfy the wedge axiom, and let $T : \widetilde{h}^* \to \widetilde{k}^*$ be a natural transformation of cohomology theories which is an isomorphism on spheres. Show that T is an isomorphism on all CW complexes.

Hint Use Problem 17.17.

17.3 The External Cohomology Product

Using the smash product, we introduce a multiplicative structure into cohomology. The external product of the classes u and v will be denoted $u \bullet v$.

Cohomology classes $u \in \widetilde{H}^n(X;G)$ and $v \in \widetilde{H}^m(Y;H)$ are homotopy classes of maps

$$u : X \to K(G,n) \qquad \text{and} \qquad v : Y \to K(H,m).$$

Then $u \bullet v \in \widetilde{H}^{n+m}(X \wedge Y; G \otimes H)$ is the map defined by the diagram

$$\begin{tikzcd}
X \wedge Y \arrow[r, "u\wedge v"] \arrow[dr, bend right, "u\bullet v"'] & K(G,n) \wedge K(H,m) \arrow[d, "c"] \\
& K(G\otimes H, n+m).
\end{tikzcd}$$

The external product defines a map (of pointed sets)

$$\widetilde{H}^n(X;G)\times\widetilde{H}^m(Y,H)\to\widetilde{H}^{n+m}(X\wedge Y;G\otimes H)$$

for every n and m. Notice that

$$|u\bullet v|=|u|+|v|.$$

The first thing to do is to establish the basic formal properties that this product.

prop:CohoProdProps

Proposition 172 *The external product defines group homomorphisms*

$$\wedge:\widetilde{H}^n(X;G)\otimes\widetilde{H}^m(Y;H)\to\widetilde{H}^{n+m}(X\wedge Y;G\otimes H).$$

These homomorphisms are (graded) commutative in the sense that[2]

$$u\bullet v=(-1)^{|u||v|}v\bullet u,$$

and associative in the sense that

$$(u\bullet v)\bullet w=u\bullet(v\bullet w).$$

Problem 17.20 Prove Proposition 172.

You'll note that if $u\in\widetilde{H}^4(X;\mathbb{Z}/a)$ and $v\in\widetilde{H}^9(X;\mathbb{Z}/a)$, for example, then there is ambiguity about the dimension of the element $u+v\in\widetilde{H}^*(X;\mathbb{Z}/a)$. For this reason, we will usually assume that each element we write down is **homogeneous**, i.e., that it is an element of $\widetilde{H}^n(X;\mathbb{Z}/a)$ for some n (in fact, we this will be a standing assumption whenever it is not explicitly stated otherwise). Since $\widetilde{H}^*(X;\mathbb{Z}/a)$ is generated by such elements, it is rarely necessary to worry about the other ones.

Although the external cohomology product was defined in terms of smash products, it is also very useful in studying the cohomology ordinary Cartesian products of spaces. In fact, since

$$X_+\wedge Y_+=(X\times Y)_+,$$

we have

$$\begin{tikzcd}
H^n(X;G)\otimes H^m(Y;H) \arrow[r, "\wedge"] \arrow[d, equal] & H^{n+m}(X\times Y;G\otimes H) \arrow[d, equal] \\
\widetilde{H}^n(X_+;G)\otimes\widetilde{H}^m(Y_+H) \arrow[r, "\wedge"] & \widetilde{H}^{n+m}(X_+\wedge Y_+;G\otimes H).
\end{tikzcd}$$

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[2] Strictly speaking, $u\bullet v\in\widetilde{H}^*(X\wedge Y;G\otimes H)=[X\wedge Y,K(G\otimes H,n+m)]$, and $v\bullet u\in[Y\wedge X,K(H\otimes G,n+m)]$. These different, but isomorphic, homotopy sets are related by the homotopy equivalences t and T discussed in the last section. Part (b) of Proposition 172 *really* means that $t_*(u\bullet v)=(-1)^{nm}T^*(v\bullet u)$.

17.4 Cohomology Rings

In this section we use the (reduced) diagonal map to convert the external product to an *internal* product; when we use ring coefficients, this endows $H^*(X)$ with the structure of a graded R-algebra.

In the special case $X = Y$, the external product gives us a map

$$\widetilde{H}^*(X; G) \otimes \widetilde{H}^*(X; H) \to \widetilde{H}^*(X \wedge X; G \otimes H).$$

Let's go one step further: from $\widetilde{H}^*(X \wedge X; G \otimes H)$ to $H^*(X; G \otimes H)$. For this we need a map $X \to X \wedge X$. Luckily, we have just such a map waiting for us. Recall that the diagonal map is $\Delta : X \to X \times X$ given by $x \mapsto (x, x)$. The **reduced diagonal map** is the map $\overline{\Delta}$ in the diagram

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr, bend right, "\overline{\Delta}"'] & X \times X \arrow[d, "\wedge"] \\ & X \wedge X, \end{tikzcd}$$

where I am using the symbol $\wedge : X \times X \to X \wedge X$ to denote the standard quotient map. Then $\overline{\Delta}^* : \widetilde{H}^*(X \wedge X) \to \widetilde{H}^*(X)$, and we can define the **cup product** of $u \in \widetilde{H}^*(X; G)$ and $v \in \widetilde{H}^*(X; H)$ to be

$$u \cdot v = \overline{\Delta}^*(u \bullet v) \in \widetilde{H}^*(X; G \otimes H).$$

In terms of a diagram, the cup product of u and v is

$$\begin{tikzcd} X \arrow[r, "\overline{\Delta}"] \arrow[drr, bend right, "u \cdot v"'] & X \wedge X \arrow[r, "u \wedge v"] & K(G, n) \wedge K(H, m) \arrow[d, "c"] \\ & & K(G \otimes H, n + m). \end{tikzcd}$$

The cup product is a homomorphism

$$\widetilde{H}^n(X; G) \otimes \widetilde{H}^m(X; H) \to \widetilde{H}^{n+m}(X; G \otimes H),$$

which is still an external product, because the coefficient groups change.

In order to get an internal product, we need the coefficient group to be a ring. If R is a ring, then the multipication map defines a function $R \otimes R \to R$, and therefore a map $m : K(R \otimes R, n) \to K(R, n)$. Therefore, we can throw this map into the mix to get an internal product

$$\begin{tikzcd}
\widetilde{H}^n(X;R)\otimes\widetilde{H}^m(X;R) \arrow[d] \arrow[rrd, bend left=10] & & \\
\widetilde{H}^{n+m}(X\wedge X;R\otimes R) \arrow[r, "m_*"] & \widetilde{H}^{n+m}(X\wedge X;R) \arrow[r, "\bar{\Delta}^*"'] & \widetilde{H}^*(X;R).
\end{tikzcd}$$

This internal cup product is given by the formula

$$u\cdot v=\bar{\Delta}^*\,m_*(u\bullet v)\in\widetilde{H}^*(X;R),$$

or, in terms of a diagram, the cup product of u and v is given by

$$\begin{tikzcd}
X \arrow[r, "\overline{\Delta}"] \arrow[rrrd, "u\cdot v"'] & X\wedge X \arrow[r, "u\wedge v"] & K(R,n)\wedge K(R,m) \arrow[r] & K(R\otimes R,n+m) \arrow[d, "r"] \\
 & & & K(R,n+m).
\end{tikzcd}$$

R-Module Structure. Let R be a ring and let X be a pointed space. Then the map $X_+\to *_+$ which collapses X to a point induces a homomorphism

$$R=H^0(*;R)\to H^0(X;R),$$

and the cup product defines a homomorphism

$$R\otimes H^n(X;R)\hookrightarrow H^0(X;R)\otimes H^n(X;R)\to H^n(X;R),$$

which we will denote by $r\otimes u\mapsto r\cdot u$.

The internal product inherits the basic algebraic properties of the external product. Thus we have the following structure theorem.

thm:CohoAlgebra

Theorem 173 *The cup product is R-bilinear, so defines a map*

$$H^*(X;R)\otimes_R H^*(X;R)\to H^*(X;R)$$

which gives $H^*(X;R)$ the structure of a graded commutative R-algebra, which is unital if R is.

PROBLEM 17.21

(a) Derive Theorem 173 from Proposition 172.

(b) Let $\widetilde{H}^*$ be a commutative graded ring, and let $x\in\widetilde{H}^n$. Show that if $n=|x|$ is odd, then $2x^2=0$.

Why Cup Product? When the internal cohomology product was first introduced, the notation for it was $u \smile v$, using the 'cup' symbol, which is why the the product is, even today, called the *cup* product.

17.5 Variation of Coefficients

In this section we study the relationships between cohomology of X with different coefficient groups (or rings).

prob:CoefficientExactSequence

PROBLEM 17.22 Show that if $0 \to A \to B \to C \to 0$ is a short exact sequence of abelian groups, then there is a long exact sequence

$$\cdots \to \widetilde{H}^n(X;A) \to \widetilde{H}^n(X;B) \to \widetilde{H}^n(X;C) \to \widetilde{H}^{n+1}(X;A) \to \cdots$$

which is natural in both X and the sequence.

Universal Coefficients Theorem. The Universal Coefficients Theorem allows us to compute the cohomology with respect to G, given the cohomology with respect to $\mathbb{Z}$. It can be generalized to compute the cohomology with respect to the R-module M given the cohomology with respect to R.

Theorem 174 (Universal Coefficients Theorem) *Suppose either*

1. *G is a finitely generated abelian group, or*
2. *X is of finite type*[3]

Then there are natural exact sequences

$$0 \to \widetilde{H}^n(X;\mathbb{Z}) \otimes G \to \widetilde{H}^n(X;G) \to \mathrm{Tor}(\widetilde{H}^{n-1}(X;\mathbb{Z}), G) \to 0$$

for each n.

PROBLEM 17.23

(a) Let $0 \to F_1 \to F_0 \to G \to 0$ be a free resolution of the group G, and use Problem 17.22 to produce a long exact sequence of cohomology groups in various coefficients. Cut the long exact sequence into a system of interlocking short exact sequences as in Problem **??**.

(b) Show that if either

   1. F is a finitely generated free abelian group, or
   2. X_{n+1} is a finite complex,

   then there is a natural isomorphism $\widetilde{H}^n(X;F) \cong \widetilde{H}^n(X;\mathbb{Z}) \otimes F$.

---

[3] i.e., X has a CW decomposition with only finitely many cells in each dimension.

(c) Use part (b) to identify the groups in the short sequences of part (a) and so prove the Universal Coefficients Theorem.

Exercise 17.24 Suppose M is an R-module, where R is a PID.

(a) Show that there is a free resolution $0 \to F_1 \to F_0 \to M \to 0$ of R-modules.

(b) Show that there are Universal Coefficient sequences with $\mathbb{Z}$ replaced with R and G replaced with M.

Problem 17.25 Suppose X is a simply-connected space with $\widetilde{H}^*(X;\mathbb{Z}) = 0$.

(a) Show that $\widetilde{H}^*(X;G) = 0$ for all abelian groups G.

(b) Show that $X \simeq *$.

Problem 17.26 Suppose $\widetilde{H}^*(X;\mathbb{Z})$ is a free abelian group. Show that $\widetilde{H}^*(X;G) \cong \widetilde{H}^*(X;\mathbb{Z}) \otimes G$.

Field Coefficients and Connectivity. Field coefficients are particularly easy to deal with – by comparison with, for example, integer coefficients – and so it is worth seeing what we can learn by studying spaces using cohomology with coefficients in a field.

Exercise 17.27 Show that an abelian group V is a rational vector space if and only if the map $V \to V$ given by multiplication by p is an isomorphism for every prime p.

prob:CohoVanishFields

Problem 17.28 Suppose $\widetilde{H}^*(X;\mathbb{Z}/p) = 0$ for all primes p.

(a) Show that $\widetilde{H}^*(X;\mathbb{Z})$ is a graded rational vector space.

(b) Show that if, in addition, $\widetilde{H}^*(X;\mathbb{Q}) = 0$ then $\widetilde{H}^*(X;\mathbb{Z}) = 0$.

Some authors interpret $\mathbb{Z}/p$ with $p = 0$ as $\mathbb{Q}$. With this convention, the conclusion of Problem 17.28 can be restated as saying that if $\widetilde{H}^*(X;\mathbb{Z}/p) = 0$ for all $p \geq 0$, then $\widetilde{H}^*(X;\mathbb{Z}) = 0$.

We can improve this result from detecting contractibility to detecting connectivity.

prop:FieldDetection

Proposition 175 *If X is simply-connected, then the following are equivalent:*

1. *$\widetilde{H}^k(X;\mathbb{Z}/p) = 0$ for $p \geq 0$ and all $k < n$*
2. *X is $(n-1)$-connected.*

Problem 17.29 Prove Proposition 175.

We can get rid of the requirement that we check every group G.

Problem 17.30

(a) Let $f : X \to Y$, where X and Y are simply connected. Assume that $f^* : \widetilde{H}^k(Y;\mathbb{Z}) \to \widetilde{H}^k(X;\mathbb{Z})$ is an isomorphism for all $k \leq n$. Show that $f^* : \widetilde{H}^k(Y;G) \to \widetilde{H}^k(X;G)$ is an isomorphism for all abelian groups G and all $k < n$.

Hint Use the fibration sequence $K(F_1, n) \to K(F_0, n) \to K(G, n)$ and the Five Lemma.

(b) Give an example of a space with $\widetilde{H}^k(X;\mathbb{Z}) = 0$ for $k \leq n$, but X only $(n-1)$-connected.

Contractible Smash Products. Now we investigate the algebraic relations between the cohomology of X and Y if $X \wedge Y \simeq *$.

Problem 17.31 Let X and Y be connected CW complexes.

(a) Show that $X \wedge Y$ is simply-connected.

(b) Suppose that for every prime $p \geq 0$, at least one of $H^*(X;\mathbb{Z}/p)$ and $H^*(Y;\mathbb{Z}/p)$ is trivial, and show that $X \wedge Y \simeq *$.

(c) On the other hand, show that if there is even one prime p for which $H^*(X;\mathbb{Z}/p)$ and $H^*(Y;\mathbb{Z}/p)$ are both nontrivial, then $X \wedge Y \not\simeq *$.

17.6 A Simple Künneth Theorem

In algebraic topology, any theorem that gives a formula for the cohomology (of whatever kind) of a product of spaces is referred to as a Künneth theorem. In this section, we prove a version of the Künneth theorem, which shows how to compute the cohomology of $X \times Y$ once you know $H^*(X;R)$ and $H^*(Y;R)$, under the assumption that $H^*(Y;R)$ is a free R-module (which is automatically true if R is a field). This is certainly a restricted statement, but it is nevertheless enourmously useful.

Throughout this section, we will write R to denote a ring with a unit element. An R-module M is **free** over R if it is isomorphic to a big direct sum of copies of R. For example, $\mathbb{Z} \oplus \mathbb{Z}$ is a free $\mathbb{Z}$-module, and $\mathbb{Z}/8 \oplus \mathbb{Z}/8$ is a free $\mathbb{Z}/8$-module.

Building Cohomology Theories. We outline two methods of building new cohomology theories from known ones. If A^* is a graded abelian group, then we write $\Sigma^m A^*$ for the graded group with $(\Sigma^m A^*)^m = A^{n-m}$.

Problem 17.32 Let $\widetilde{h}^*$ and $\widetilde{k}^*$ be cohomology theories. Is this true for infinite sums of cohomology theories?

(a) Show that $\widetilde{h}^* \oplus \widetilde{k}^*$ is also a cohomology theory.

(b) Show that, for fixed m, $\widetilde{\ell}^n(\,?\,) = \widetilde{h}^{n-m}(\,?\,)$ is also a cohomology theory.

(c) Suppose $\widetilde{h}^*$ is a cohomology theory which takes its values in the category of graded R-modules. Show that for any free graded R-module A^*, $\widetilde{h}^*(\,?\,) \otimes_R A^*$ is also a cohomology theory.

The Cohomology of a Product. We know that

$$\widetilde{H}^*(S^n; R) = \begin{cases} R & \text{if } * = n \\ 0 & \text{otherwise,} \end{cases}$$

which is a free graded R-module in any case. We also have isomorphisms

$$\widetilde{H}^k(X;R) \to R \otimes_R \widetilde{H}^k(X;R) \quad \text{and} \quad \widetilde{H}^k(X;R) \to \widetilde{H}^{k+1}(\Sigma X;R).$$

Problem 17.33 By going back through the definitions, show that the diagram

$$\begin{tikzcd}
\widetilde{H}^*(X;R) \arrow[d, "\cong"'] \arrow[rr, bend left=10, "\cong"] & & \\
R \otimes_R \widetilde{H}^k(X;R) \arrow[r, "\cong"] & \widetilde{H}^1(S^1;R) \otimes_R \widetilde{H}^k(X;R) \arrow[r] & \widetilde{H}^{k+1}(\Sigma X;R)
\end{tikzcd}$$

is commutative. Conclude that in general, the external product

$$\widetilde{H}^*(S^n;R) \otimes_R \widetilde{H}^*(X;R) \to \widetilde{H}^*(\Sigma^n X;R)$$

can be identified with the suspension isomorphism.[4]

If M is a free R-module and $A \to B \to C$ is an exact sequence of R-modules, then $A \otimes_R M \to B \otimes_R M \to C \otimes_R M$ is also exact. When R is a field, then all modules are free, and tensor product is particularly nice and friendly.

Exercise 17.34 Concoct an example of an exact sequence of finite abelian groups $A \to B \to C$ and another finite abelian group M with the property that $A \otimes M \to B \otimes M \to C \otimes M$ is not exact.

Finally, we should be more precise about the tensor products of graded abelian groups and graded rings. So let's say H^* and K^* are two graded abelian groups. Then we would like to say that $H^* \otimes K^*$ is also a graded abelian group, and here's how we do it:

$$(H^* \otimes K^*)^n = \bigoplus_{i+j=n} H^i \otimes K^j.$$

---

[4] It is possible that a ±1 will emerge when you work through this problem; be careful, but don't worry.

This should remind you of the way the i-cells in X and the j-cells in Y give rise to n-cells in $X \wedge Y$. If H^* and K^* are graded rings, then we define the multiplication on $H^* \otimes K^*$ by the rule

$$(h_1 \otimes k_1) \cdot (h_2 \otimes k_2) = (-1)^{|k_1||h_2|}(h_1h_2 \otimes k_1k_2).$$

PROBLEM 17.35 Verify that the external product defines a homomorphism of graded rings

$$\widetilde{H}^*(X;R) \otimes_R \widetilde{H}^*(Y;R) \to \widetilde{H}^*(X \wedge Y;R).$$

Our goal is to show that this map is actually an **isomorphism** of graded rings in many cases, including the case where R is a field.

thm:Kunneth1

Theorem 176 *Let X and Y be CW complexes, and assume that $\widetilde{H}^*(Y;R)$ is a free R-module. Then the external product defines an isomorphism of graded rings*

$$\widetilde{H}^*(X;R) \otimes_R \widetilde{H}^*(Y;R) \to \widetilde{H}^*(X \wedge Y;R).$$

PROBLEM 17.36

(a) Let Y be any space, and define functors $K^n(\,?\,) = \widetilde{H}^n(\,?\wedge Y;R)$ Show that the functors K^n constitute a cohomology theory.

(b) Suppose that $\widetilde{H}^*(Y;R)$ is a free R-module, and define

$$L^*(\,?\,) = \widetilde{H}^*(X;R) \otimes_R \widetilde{H}^*(Y;R)$$

Show that the functors L^n constitute a cohomology theory.

(c) Show that the external cohomology product defines a natural transformation $\Phi : L^* \to K^*$; and show that $\Phi : L^*(S^n) \to K^*(S^n)$ is an isomorphism for each n.

(d) Prove Theorem 176.

We immediately derive the unpointed version, which tells us about the cohomology of a product.

cor:Kunneth2

Corollary 177 *Let X and Y be CW complexes, and assume that $\widetilde{H}^*(Y;R)$ is a free R-module. Then the external product defines an isomorphism of graded rings*

$$H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R).$$

Let's do some computations.

Products of Spheres. We know that $H^*(S^n; R) = \Lambda_R(x_n)$, an exterior algebra on a unit element $1 \in H^0(S^n; R)$ and $x_n \in H^n(S^n; R)$, subject to the single relation $x_n^2 = 0$.

prob:ProdSphereCo

Problem 17.37 s

(a) Give a complete description of $H^*(S^{n_1} \times S^{n_2}; R)$.

(b) More generally, describe $H^*(\prod_{i=1}^k S^{n_i}; R)$.

(c) Let $P = \overbrace{S^n \times S^n \times \cdots \times S^n}^{k \text{ factors}}$, and let $u_1, u_2, \ldots, u_k \in H^n(P; R)$ be the generators corresponding to the various spheres. Evaluate

$$(u_1 + u_2 + \cdots + u_k)^k.$$

Cohomology of the Loop Space of a Sphere. In Problem **??** you used the James splitting to compute the cohomology groups of ΩS^{n+1}. Since the James construction relates ΩS^{n+1} to products of spheres, whose cohomology algebras we understand, we can make a study of cup products in $H^*(\Omega S^{n+1}; R)$.

Choose generators $y_k \in H^{nk}(\Omega S^{n+1}; R) \cong R$. Then $y_k \cdot x_l = c_{k,l}\, y_{k+l}$ for some coefficient $c_{k,l} \in R$. How can we determine these coefficients?

prob:JamesData

Problem 17.38

(a) Show that there is a (homotopy) pushout square

$$\begin{tikzcd}
T^{k-1}(S^n) \arrow[r] \arrow[d, "\text{in}"'] & J^{k-1}(S^n) \arrow[d] \\
S^n \times \cdots \times S^n \arrow[r, "q_k"] & J^k(S^n)
\end{tikzcd}$$

(b) Show that the map $S^n \times \cdots \times S^n \to J^k(S^n)$ induces an isomorphism in $H^{kn}(\,?\,)$.

Hint Take cofibers of the vertical maps.

(c) Show that $q_k^* : H^{nk}(J^k(S^n); R) \to H^{nk}(S^n \times \cdots \times S^n; R)$ is an isomorphism.

(d) Show that $q_1^* : H^n(J(S^n); R) \to H^n(S^n \times \cdots \times S^n; R)$ is given by $y_1 \mapsto x_1 + \cdots + x_k$.

prob:CohoLoopSpheres1

Problem 17.39 Let n be even. We want to determine any and all algebraic relations among the various generators $y_i \in H^{ni}(J^k(S^n); \mathbb{Z}) \cong \mathbb{Z}$.

(a) Because $H^{nk}(\Omega S^{n+1}; \mathbb{Z}) \cong \mathbb{Z} \cdot y_k$, we know that $y^k = c_k\, y_k$, for some coefficient $c_k \in \mathbb{Z}$. Determine the integer c_k.[5]

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[5] Up to a sign – you can, if you like, go back and redefine the generators y_k so that each of the coefficients c_k is ≥ 0.

(b) Use part (c) to determine the coefficient $c_{k,l}$ in the equation

$$y_k \cdot y_l = c_{k,l}\, y_{k+l}.$$

(c) Summarize what you have learned by giving a complete description of the ring structure on $H^*(\Omega S^{n+1}; \mathbb{Z})$.

Chapter 18

Projective Spaces

Projective spaces (for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$) are among the most important examples in algebraic topology, coming in right after spheres. In this chapter we're going to take a careful look at projective spaces.

Interpret as space of lines.

18.1 Definitions

Write $\mathbb{F}$ to denote either of the fields $\mathbb{R}$, $\mathbb{C}$ or the skew field[1] $\mathbb{H}$ ($\mathbb{H}$ denotes the **quaternions**, and it is $\mathbb{R}^4$ with a certain noncommutative multiplication).

Let d be the dimension of $\mathbb{F}$, so d is either $1, 2$ or 4. Then the set of all $x \in \mathbb{F}$ with $|x| = 1$ is the standard unit sphere S^{d-1}. The multiplication in $\mathbb{F}$ respects length, i.e.,

$$|gx| = |g| \cdot |x| \quad \text{for all} \quad g \in \mathbb{F}, x \in \mathbb{F}^{n+1}.$$

Therefore S^{d-1} is a group under multiplication, and it acts by multiplication on $\mathbb{F}^{n+1}$ for all n. Inside of $\mathbb{F}^{n+1}$ is the unit sphere

$$S^{nd+(d-1)} = \{x \in \mathbb{F}^{n+1} \mid |x| = 1\}.$$

If $g \in S^{d-1}$ and $x \in S^{nd+(d-1)}$ then $g \cdot x \in S^{nd+(d-1)}$ too, which means that the group S^{d-1} acts on the space $S^{nd+(d-1)}$.

---

[1] A **skew field** is a ring in which every element has a multiplicative inverse, but, unlike an ordinary field, the multiplication is not required to be commutative.

Definition 178 The n^{th} **projective space** for $\mathbb{F}$ is the orbit space[2]

$$\mathbb{F}\mathrm{P}^n = S^{nd+(d-1)}/\big(\text{action of } S^{d-1}\big).$$

We will write $[x_1, \ldots, x_{n+1}] \in \mathbb{F}\mathrm{P}^n$ to denote the equivalence class of $(x_1, \ldots, x_{n+1}) \in S^{nd+(d-1)}$. There is, of course, a canonical identification map

$$q_n : S^{nd+(d-1)} \to \mathbb{F}\mathrm{P}^n$$

given by the formula $(x_1, \ldots, x_{n+1}) \mapsto [x_1, \ldots, x_{n+1}]$. The description of points in $\mathbb{F}\mathrm{P}^n$ in this way is called representation by **homogeneous coordinates**.

EXERCISE 18.1 Show that $\mathbb{F}\mathrm{P}^0 = *$.

The inclusions $\mathbb{F}^n \to \mathbb{F}^{n+1}$ given by $x \mapsto (x, 0)$ respect the action of S^{d-1}, and so they give rise to commutative diagrams:

$$\begin{tikzcd}
S^{nd-1} \arrow[r] \arrow[d, "q_{n-1}"'] & S^{(n+1)d-1} \arrow[d, "q_n"] \\
\mathbb{F}\mathrm{P}^{n-1} \arrow[r] & \mathbb{F}\mathrm{P}^n
\end{tikzcd}$$

where the horizontal maps just take $x \mapsto (x, 0)$ and $[x] \to [x, 0]$. These maps fit into a commutative ladder

$$\begin{tikzcd}
S^{d-1} \arrow[r] \arrow[d, "q_0"'] & S^{2d-1} \arrow[r] \arrow[d, "q_n"] & \cdots \arrow[r] & S^{nd-1} \arrow[r] \arrow[d, "q_{n-1}"'] & S^{(n+1)d-1} \arrow[r] \arrow[d, "q_n"] & \cdots \\
\mathbb{F}\mathrm{P}^0 \arrow[r] & \mathbb{F}\mathrm{P}^1 \arrow[r] & \cdots \arrow[r] & \mathbb{F}\mathrm{P}^{n-1} \arrow[r] & \mathbb{F}\mathrm{P}^n \arrow[r] & \cdots
\end{tikzcd}$$

The two horizontal maps in this are cofibrations (as we will see in the next section), and so the colimits of the rows

$$S^\infty = \bigcup S^{nd-1} \quad \text{and} \quad \mathbb{F}\mathrm{P}^\infty = \bigcup \mathbb{F}\mathrm{P}^n,$$

are also *homotopy* colimts. Alternatively,

$$\mathbb{F}\mathrm{P}^\infty = S^\infty/(\text{action of } S^{d-1}).$$

The projective spaces are extremely important examples in homotopy theory, so we are going to study them in some detail in this chapter.

---

[2] That is, we can define an equivalence relation on $S^{nd+(d-1)}$ by setting $x \sim y$ if there is a $g \in S^{d-1}$ such that $g \cdot x = y$; the orbit space is the set of $\sim$-equivalence classes $S^{nd+(d-1)}/\sim$.

Theorem 179 *For each $\mathbb{F}$ and each $0 \le n \le \infty$, the map*

$$q_n : S^{nd+(d-1)} \to \mathbb{F}\mathrm{P}^n$$

is a fibration whose fiber is S^{d-1}.

PROBLEM 18.2 Prove Theorem 179.

HINT You've already done the case $n < \infty$.

PROBLEM 18.3 Show that the square

$$\begin{tikzcd} S^{nd+(d-1)} \arrow[r] \arrow[d, "q_n"'] & S^{\infty} \arrow[d, "q_\infty"] \\ \mathbb{F}\mathrm{P}^n \arrow[r] & \mathbb{F}\mathrm{P}^{\infty} \end{tikzcd}$$

is a homotopy pullback square.

PROBLEM 18.4

(a) Show that $S^\infty \simeq *$.

(b) Determine the homotopy fiber of $\mathbb{F}\mathrm{P}^n \hookrightarrow \mathbb{F}\mathrm{P}^\infty$?

(c) Show that $\Omega\mathbb{F}\mathrm{P}^\infty \simeq S^{d-1}$.

(d) Show that $\mathbb{R}\mathrm{P}^\infty = K(\mathbb{Z}/2, 1)$.

(e) Show that $\mathbb{C}\mathrm{P}^\infty = K(\mathbb{Z}, 2)$.

(f) What can you say about $\mathbb{H}\mathrm{P}^\infty$?

PROBLEM 18.5 Show that $\mathbb{C}\mathrm{P}^n \to \mathbb{C}\mathrm{P}^\infty$ generates $H^2(\mathbb{C}\mathrm{P}^n; \mathbb{Z})$ and $\mathbb{R}\mathrm{P}^n \to \mathbb{R}\mathrm{P}^\infty$ generates $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$.

For the finite-dimensional projective spaces, we can still express their homotopy groups in terms of the homotopy groups of spheres. This is because the fibration sequence $S^{d-1} \to S^{nd+(d-1)} \to \mathbb{F}\mathrm{P}^n$ splits after it is looped once.

PROBLEM 18.6 Show that for each $n \ge 1$, $\Omega(\mathbb{F}\mathrm{P}^n) \simeq S^{d-1} \times \Omega(S^{nd+(d-1)})$.

These loop space splittings tell us something about the homotopy groups of projective spaces.

Corollary 180

(a) $\pi_k(\mathbb{F}\mathrm{P}^n) \cong \pi_{k-1}(S^{d-1}) \oplus \pi_k(S^{nd+(d-1)})$ *for* $\mathbb{F} = \mathbb{C}$ *or* $\mathbb{H}$.

(b) $\pi_k(\mathbb{R}\mathrm{P}^n) \cong \pi_k(S^n)$ *for* $k > 1$

(c) $\pi_1(\mathbb{R}\mathrm{P}^n) \cong \mathbb{Z}/2$.

PROBLEM 18.7 Prove Proposition **??** and Corollary 180 using Problem 14.19. What does Corollary 180 say about the homotopy groups of spheres in the case $n = 1$?

PROBLEM 18.8 Let $i_n : \mathbb{F}\mathrm{P}^n \hookrightarrow \mathbb{F}\mathrm{P}^\infty$. Show that

$$\begin{tikzcd} S^{nd+(d-1)} \arrow[r, "q_n"] & \mathbb{F}\mathrm{P}^n \arrow[r, "i_n"] & \mathbb{F}\mathrm{P}^\infty \end{tikzcd}$$

is a fibration sequence.

HINT Let $F \to \mathbb{F}\mathrm{P}^n$ be the fiber of i_n and construct a commutative triangle

$$\begin{tikzcd} S^{nd+(d-1)} \arrow[d, "\xi"'] \arrow[dr, "q_n"] & \\ F \arrow[r] & \mathbb{F}\mathrm{P}^n. \end{tikzcd}$$

Show that ξ induces isomorphisms on π_*.

18.2 Cellular Decomposition of $\mathbb{F}\mathrm{P}^n$

We will show that the projective spaces are CW complexes by explicitly working out extremely efficient CW decompositions for them.

Inside of $S^{nd+(d-1)} \subseteq \mathbb{F}^{n+1}$ is the subset

$$E = \{(x_1, \dots, x_{n+1}) \in S^{nd+(d-1)} \mid x_{n+1} \in \mathbb{R}_{\geq 0}\}.$$

PROBLEM 18.9 Show that each $x \in S^{nd+(d-1)}$ with $x_k \neq 0$ is equivalent to a unique element of $S^{nd+(d-1)}$ with $x_k \in \mathbb{R}_{\geq 0}$.

PROBLEM 18.10 Let $D^{nd} \subseteq \mathbb{F}^n$ be the standard nd-dimensional disk, and define $f : D^{dn} \to E$ by the formula $f(x) = (x, \sqrt{1 - |x|^2})$.

(a) Show that f is a homeomorphism.

(b) Show that the boundary of E is the standard $(nd-1)$-sphere $S^{dn-1} \subseteq \mathbb{F}^n \times 0 \subseteq \mathbb{F}^{n+1}$.

PROBLEM 18.11 Referring to the action of S^{d-1} on $S^{nd+(d-1)}$, let's say $x \sim y$ if and only if x and y are in the same orbit.

(a) Show that every $x \in S^{nd+(d-1)}$ is $\sim$-equivalent to a point in E; conclude that the quotient map $q_n|_E : E \to \mathbb{F}\mathrm{P}^n$ is surjective.

(b) Show that if $x = (x_1, \dots, x_{n+1})$ with $x_{n+1} \neq 0$, then x is equivalent to a **unique** point in the interior of E; conclude that $q_n|_{\mathrm{int}(E)}$ is injective.

These problems contain all the hard work in the construction of our CW decompositions of $\mathbb{F}\mathrm{P}^n$.

Theorem 181 *For every* n, $\mathbb{F}\mathrm{P}^n = \mathbb{F}\mathrm{P}^{n-1} \cup_{q_{n-1}} D^{nd}$.

Problem 18.12 Prove Theorem 181.

This gives us the inductive step in the CW decomposition of $\mathbb{F}\mathrm{P}^n$. Here is the conclusion.

Corollary 182 *For each* n, $\mathbb{F}\mathrm{P}^n$ *has a CW decomposition of the form*

$$\mathbb{F}\mathrm{P}^n = * \cup D^d \cup D^{2d} \cup D^{3d} \cup \cdots \cup D^{nd}.$$

Corollary 183 *If* $m \leq n$, *then* $\mathbb{F}\mathrm{P}^m$ *is a subcomplex of* $\mathbb{F}\mathrm{P}^n$.

Corollary 184 *If* $m \leq n$, *then the inclusion* $\mathbb{F}\mathrm{P}^m \hookrightarrow \mathbb{F}\mathrm{P}^n$ *is a cofibration.*

Problem 18.13

(a) What is $\mathbb{F}\mathrm{P}^1$?

(b) Use the fibration sequence $S^{d-1} \to S^{2d-1} \to \mathbb{F}\mathrm{P}^1$ to learn something new about the higher homotopy groups of some spheres.

As you know well, as soon as you know the cellular structure of a space, you can get some information about its cohomology.

Problem 18.14

(a) Determine the cohomology (with coefficient group $\mathbb{Z}$) of $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$.

(b) Can you do the same for $\mathbb{R}\mathrm{P}^n$? Explain.

(c) How does your answer in (a) change if the coefficient group is an arbitrary group abelian G?

18.3 Collapse Maps for $\mathbb{R}\mathrm{P}^n$

When it comes to calculating the cohomology of real projective spaces, we need to work by induction on our **cone decomposition**:

$$\begin{tikzcd}
S^0 \arrow[d, "q_0"'] & S^1 \arrow[d, "q_1"'] & S^2 \arrow[d, "q_2"'] & & S^{n-1} \arrow[d, "q_{n-1}"'] & \\
\mathbb{R}\mathrm{P}^0 \arrow[r] & \mathbb{R}\mathrm{P}^1 \arrow[r] & \mathbb{R}\mathrm{P}^2 \arrow[r] & \cdots \arrow[r] & \mathbb{R}\mathrm{P}^{n-1} \arrow[r] & \mathbb{R}\mathrm{P}^n.
\end{tikzcd}$$

The resulting exact sequences don't simply fall apart for us, so we'll need to determine some of the maps in the sequence. Start by writing our cone decomposition of $\mathbb{R}\mathrm{P}^n$ vertically in the left column of the following diagram. Then extend the 'L-shaped' cofiber sequences to long zig-zag cofiber sequences, and weave them together, resulting in the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
S^{n-1} \arrow[r, dotted] & \mathbb{R}\mathrm{P}^{n-1} \arrow[r] \arrow[d, dotted] & S^{n-1} \arrow[r] & \Sigma\mathbb{R}\mathrm{P}^{n-2} \arrow[r] \arrow[d] & S^{n-1} \arrow[r] & \cdots \\
S^{n} \arrow[r, "q_n"'] \arrow[rr, dashed, bend left, "\delta_n"] & \mathbb{R}\mathrm{P}^{n} \arrow[r, dotted, "j_n"'] \arrow[d] & S^{n} \arrow[r, dotted, "\Sigma q_{n-1}"'] \arrow[rr, dashed, bend left, "\Sigma\delta_{n-1}"] & \Sigma\mathbb{R}\mathrm{P}^{n-1} \arrow[r, "\Sigma j_{n-1}"'] \arrow[d, dotted] & S^{n} \arrow[r] & \cdots \\
S^{n+1} \arrow[r] & \mathbb{R}\mathrm{P}^{n+1} \arrow[r] \arrow[d] & S^{n+1} \arrow[r] & \Sigma\mathbb{R}\mathrm{P}^{n} \arrow[r, dotted] \arrow[d] & S^{n+1} \arrow[r, dotted] & \cdots \\
 & \vdots & & \vdots & &
\end{tikzcd}$$

in which the dotted zig-zag is cofiber sequence mentioned above, and the other zig-zags are the cofiber sequences for other values of n.

Lemma 185 *The map* $\delta_n = j_n \circ q_n : S^n \to S^n$ *has degree* $1 + (-1)^{n+1}$. *That is,* $\deg(\delta_n) = 0$ *if* n *is even, and* $\deg(\delta_n) = 2$ *if* n *is odd.*[3]

PROBLEM 18.15 Prove Lemma 185 as follows.

(a) Show that there a map $\beta : S^n \vee S^n \to S^n$ so that the solid arrows in

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d] & \mathbb{R}\mathrm{P}^{n-1} \arrow[d] \\
S^{n} \arrow[r, "q_n"] \arrow[d, "\phi"'] \arrow[dr, dotted, "\delta_n"] & \mathbb{R}\mathrm{P}^{n} \arrow[d, "j_n"] \\
S^n \vee S^n \arrow[r, "\beta"] & S^n
\end{tikzcd}$$

form a strictly commutative diagram, where the vertical maps are cofiber sequences and so ϕ is the pinch map that we used to define the group structure on $\pi_n(X)$.

(b) Since $\beta \in [S^n \vee S^n, S^n] \cong [S^n, S^n] \times [S^n, S^n]$, we can write $\beta = (\beta_1, \beta_2)$. Show that $\delta_n = [\beta_1] + [\beta_2]$.

(c) Show that β_1 and β_2 are homeomorphisms (it is enough to show that they are bijective). Conclude that $|\deg(\beta_1)| = |\deg(\beta_2)| = 1$.

---

[3] If we choose our generator for $H^n(S^n; \mathbb{Z})$ differently, then the map will have degree 0 or -2, respectively, but this has no practical effect on our applications.

(d) Show that $\beta_2 = a \circ \beta_1$, where $a : S^n \to S^n$ is the antipodal map. Conclude that $\deg(\beta_2) = (-1)^{n+1} \deg(\beta_1)$, and hence

$$\deg(\delta_n) = \pm(1 + (-1)^{n+1}).$$

Hint Think of S^n as $\Sigma_0 S^{n-1}$ (the unreduced suspension of S^{n-1}), so a point of S^n has the form $[x, t]$, where $x \in S^{n-1}$ and $t \in I$. What is the antipode of the point $[x, t]$?

Now choose a coefficient group G, and apply the functor $H^n(\,?\,; G)$ to the big diagram of spaces we constructed above to obtain

$$\begin{tikzcd}
& \vdots & & \vdots & \\
0 & 0 \arrow[l, dotted] \arrow[u] & 0 \arrow[l] & 0 \arrow[l] \arrow[u] & 0 \arrow[l] \\
H^n(S^n) & H^n(\mathbb{R}\mathrm{P}^n) \arrow[l, "q_n^*"'] \arrow[u, dotted] & H^n(S^n) \arrow[l, dotted, "j_n^*"'] \arrow[ll, dashed, bend right, "\delta_n^* = 0 \text{ or } 2"'] & H^n(\Sigma\mathbb{R}\mathrm{P}^n) \arrow[l, dotted, "q_{n-1}^*"'] \arrow[u] & H^n(\Sigma S^{n-1}) \arrow[l, "j_{n-1}^*"'] \arrow[ll, dashed, bend right, "\delta_{n-1}^* = 2 \text{ or } 0"'] \\
H^n(S^{n+1}) & H^n(\mathbb{R}\mathrm{P}^{n+1}) \arrow[l] \arrow[u] & 0 \arrow[l] & H^n(\Sigma\mathbb{R}\mathrm{P}^n) \arrow[l] \arrow[u, dotted] & 0 \arrow[l, dotted] \\
& \vdots \arrow[u] & & \vdots \arrow[u] &
\end{tikzcd}$$

(Why are we justified in labeling the second curved arrow δ_{n-1}^* instead of $(\Sigma\delta_{n-1})^*$?) There is a lot of information hidden in this diagram. Your next problem is to tease out some of it.

Problem 18.16

(a) Show that $j_n^* : H^n(S^n; G) \to H^n(\mathbb{R}\mathrm{P}^n; G)$ is surjective. Conclude that $H^n(\mathbb{R}\mathrm{P}^n) \cong \frac{H^n(S^n)}{\ker(j_n^*)}$

(b) Show that $\mathrm{Im}(q_n^*) = \mathrm{Im}(\delta_n^*)$ and $\mathrm{Im}(q_{n-1}^*) = \mathrm{Im}(\delta_{n-1}^*)$.

(c) Determine $H^n(\mathbb{R}\mathrm{P}^n; G)$ in terms of G; your answer will depend on whether n is even or odd.

Problem 18.17 Here are some more useful bits of information about this diagram

(a) Show that $H^n(\mathbb{R}\mathrm{P}^{n+1}; G) \to H^n(\mathbb{R}\mathrm{P}^n; G)$ is injective.

(b) Show that q_n^* is either the zero map or multiplication by 2.

(c) If $G = \mathbb{Z}/2$, show that j_n^* is injective, no matter what n is.

Here is the promised calculation.

Proposition 186 *The cohomology of* $\mathbb{R}\mathrm{P}^n$ *is given by*

$$H^k(\mathbb{R}\mathrm{P}^n; G) \quad = \quad \begin{cases} G & \text{if } k = 0 \\ G \otimes \mathbb{Z}/2 & \text{if } 1 \le k < n, \ k \text{ even} \\ \mathrm{Tor}(G, \mathbb{Z}/2) & \text{if } 1 \le k < n, \ k \text{ odd} \end{cases}$$

$$H^n(\mathbb{R}\mathrm{P}^n; G) \quad = \quad \begin{cases} G & \text{if } n \text{ is odd} \\ G \otimes \mathbb{Z}/2 & \text{if } n \text{ is even.} \end{cases}$$

Problem 18.18 Prove Proposition **??**.

Hint Use induction.

Exercise 18.19 Work out $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z})$ and $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$.

We will come back to this approach to calculation in much greater generality in later chapters. The following exercise hints at the broader importance of our work in this section.

Exercise 18.20 Referring to the large diagram of cohomology groups we constructed,

(a) show that $\delta_n^* \circ \delta_{n-1}^* = 0$, so that $\mathrm{Im}(\delta_{n-1}^*) \subseteq \ker(\delta_n^*)$;

(b) show that $H^k(\mathbb{R}\mathrm{P}^{n+1}) \cong \frac{\ker(\delta_n^*)}{\mathrm{Im}(\delta_{n-1}^*)}$.

18.4 The Diagonal Map in $\mathbb{F}\mathrm{P}^n$

We have successfully determined the cohomology *groups* of the projective spaces, but what about their ring structure? Our next goal is to determine the cohomology ring structure for $\mathbb{F}\mathrm{P}^n$. Since cup products in $H^*(X)$ are defined in terms of the diagonal map $\Delta : X \to X \times X$, we need to study the diagonal map in $\mathbb{F}\mathrm{P}^n$.

We begin by studying the diagonal map for a disk. It will simplify our work considerably if, instead of using the standard round disk, we can instead work with cubes, so our first job is to establish the required notation. Write $J_{\mathbb{R}} = D^1 = [-1, 1]$. Then we have a homeomorphism

$$\sigma : D^n \to J_{\mathbb{R}}^n$$

given by radial scaling[4]. Now S^{n-1} maps homeomorphically to boundary of $J^n_{\mathbb{R}}$, which is

$$S^{n-1} \cong \{x \in J^n_{\mathbb{R}} \,|\, x_i = \pm 1 \text{ for at least one } i\}.$$

This homeomorphism has the property that $\sigma(-x) = -\sigma(x)$ for all $x \in D^n$, i.e., it *respects the antipodal action.*

Now fix $s, t \in \mathbb{N}$ such that $s + t = n$. Then $J^n_{\mathbb{R}} = J^s_{\mathbb{R}} \times J^t_{\mathbb{R}}$, and we can write elements of $J^n_{\mathbb{R}}$ in the form (x, y) with $x \in J^s_{\mathbb{R}}$ and $y \in J^t_{\mathbb{R}}$. For clarity later, we'll let $0^s \in \mathbb{R}^s$ and $0^t \in \mathbb{R}^t$ denote the origins.

Some Notation. It is customary to define the **product** of pairs (X, A) and (Y, B) to be the pair $(X \times Y, X \times B \cup A \times Y)$. However, as you can easily check, this is not the categorical product in the category of pairs.[5] Actually, the importance of this construction is that it fits into the exponential law. In the study of category theory constructions like this have come to be denoted by $\otimes$, since the tensor product plays the same role in an exponential law for R-modules. But this seems to be a bit of a stretch for students just learning about tensor products, and because it applies a typically algebraic notation to spaces. So I'm going to use some new notation for the product pair:

$$(X, A) \boxplus (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

This notation was handy, not in use for anything that I have seen, and evokes the standard picture of the location of the subspace $X \times B \cup A \times Y$ inside of $X \times Y$.

Now let us consider the diagram of pairs

$$\begin{tikzcd}
(J^n_{\mathbb{R}}, *) \arrow[r, "\Delta"] \arrow[d, equal] & (J^n_{\mathbb{R}}, *) \boxplus (J^n_{\mathbb{R}}, *) \arrow[d, "u"] \\
(J^n_{\mathbb{R}}, *) \arrow[r] \arrow[d, "r"'] & (J^n_{\mathbb{R}}, S^{s-1} \times 0^t) \boxplus (J^n_{\mathbb{R}}, 0^s \times S^{t-1}) \\
(J^n_{\mathbb{R}}, S^{n-1}) \arrow[r, dotted, "\xi"] & (J^s_{\mathbb{R}} \times 0^t, S^{s-1} \times 0^t) \boxplus (0^s \times J^t_{\mathbb{R}}, 0^s \times S^{t-1}) \arrow[u, "i"']
\end{tikzcd}$$

where $J^s_{\mathbb{R}}$ is identified with the first s coordinates, and $J^t_{\mathbb{R}}$ with the last t. Our goal is to show that there is a homeomorphism ξ which renders the

---

[4] Write down the formula!

[5] Exercise: Check it! What is the product in the category of pairs?

diagram commutative *up to a homotopy of pairs.* In fact, it is easy to define this map and homotopy; here you go:

$$\xi(x,y) = ((x,0),(0,y)) \quad \text{and} \quad H((x,y),t) = ((x,ty),(tx,y)).$$

Problem 18.21

(a) Show that $H : u \circ \Delta \simeq i \circ \xi \circ r$.

(b) Show that H is a homotopy of pairs.

(c) Show that H respects the antipodal action.

Now we can use our homeomorphism σ to make the same conclusions about the ordinary round disks.

lem:RPnDiagonalHomotopy

Lemma 187 *There is map $\zeta : (D^n, S^{n-1}) \to (D^s, S^{s-1}) \times (D^t, S^{t-1})$ such that in the diagram*

$$\begin{tikzcd}
(D^n,*) \arrow[r, "\Delta"] \arrow[d, equal] & (D^n,*) \boxplus (D^n,*) \arrow[d] \\
(D^n,*) \arrow[r, "d"] \arrow[d, "r"'] & (D^n, S^{s-1}) \boxplus (D^n, S^{t-1}) \\
(D^n, S^{n-1}) \arrow[r, dotted, "\zeta"] & (D^s, S^{s-1}) \boxplus (D^t, S^{t-1}) \arrow[u, "i"']
\end{tikzcd}$$

there is a homotopy $K : i \circ \zeta \circ r \simeq d$ such that

1. *K is a homotopy of pairs,*
2. *ζ is a homeomorphism of pairs, and*
3. *For each t, $K(-x,t) = -K(x,t)$.*

Exercise 18.22 Prove Lemma 187 in full detail.

Now consider the characteristic maps $D^n \to \mathbb{R}\mathrm{P}^n$ that we discussed in the construction of CW decompositions for projective spaces. If we apply them to each disk in the diagram of Lemma 187, we obtain the following diagram of pairs

$$\begin{tikzcd}
(\mathbb{R}\mathrm{P}^n,*) \arrow[r, "\Delta"] \arrow[d] & (\mathbb{R}\mathrm{P}^n,*) \boxplus (\mathbb{R}\mathrm{P}^n,*) \arrow[d] \\
(\mathbb{R}\mathrm{P}^n,*) \arrow[r] \arrow[d] & (\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{s-1}) \boxplus (\mathbb{R}\mathrm{P}^n, \widehat{\mathbb{R}\mathrm{P}}^{t-1}) \\
(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^{n-1}) \arrow[r] & (\mathbb{R}\mathrm{P}^s, \mathbb{R}\mathrm{P}^{s-1}) \boxplus (\widehat{\mathbb{R}\mathrm{P}}^t, \widehat{\mathbb{R}\mathrm{P}}^{t-1}) \arrow[u]
\end{tikzcd}$$

which is commutative up to a homotopy of pairs, because the homotopy of Lemma 187 respects the S^0 action.[6] Now collapse the pairs to get

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^n \arrow[r, "\bar{\Delta}"] \arrow[d, equal] & \mathbb{R}\mathrm{P}^n \wedge \mathbb{R}\mathrm{P}^n \arrow[d] \\
\mathbb{R}\mathrm{P}^n \arrow[r] \arrow[d, "j_n"'] & (\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{s-1}) \wedge (\mathbb{R}\mathrm{P}^n/\widehat{\mathbb{R}\mathrm{P}}^{t-1}) \\
S^n \arrow[r, "\cong"] & S^n. \arrow[u]
\end{tikzcd}$$

The commutativity of this diagram is the key to our understanding of the cup product in $H^*(\mathbb{R}\mathrm{P}^n)$.

prob:RPnZ2stuff

Problem 18.23 Using $\mathbb{Z}/2$ coefficients, show that

(a) $j_n^* : H^n(S^n) \to H^n(\mathbb{R}\mathrm{P}^n)$ is an isomorphism.

(b) Let $n \leq m$. Show that the inclusion $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{R}\mathrm{P}^m$ induces isomorphisms

$$H^k(\mathbb{R}\mathrm{P}^m) \to H^k(\mathbb{R}\mathrm{P}^n) \quad \text{for } k \leq n.$$

Theorem 188 $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \cong (\mathbb{Z}/2)[x_1]/(x_1^{n+1})$.

We know that $H^k(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) \cong \mathbb{Z}/2$ for $0 \leq k \leq n$; let x_k be the unique nonzero element of this group.

To prove the theorem, we just have to show that $x_s x_t = x_{s+t}$. Because of the Problem 18.23, it suffices to verify this in the cohomology of $\mathbb{R}\mathrm{P}^n$, where $n = s + t$.

Problem 18.24 Write $K_s = K(\mathbb{Z}/2, s)$ and $K_t = K(\mathbb{Z}/2, t)$, so that x_s and x_t are homotopy classes of maps $x_s : \mathbb{R}\mathrm{P}^n \to K_s$ and $x_t : \mathbb{R}\mathrm{P}^n \to K_t$.

(a) Show that x_s factors through a map $\bar{x}_s : \mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{s-1} \to K_s$ and x_t factors through a map $\bar{x}_t : \mathbb{R}\mathrm{P}^n/\widehat{\mathbb{R}\mathrm{P}}^{t-1} \to K_t$.

---

[6] Remember that $\mathbb{R}\mathrm{P}^{t-1} \subseteq \mathbb{R}\mathrm{P}^n$ is the set of points with the form $[x_1, x_2, \ldots, x_t, 0, 0, \ldots, 0]$; the notation $\widehat{\mathbb{R}\mathrm{P}}^{t-1}$ is supposed to indicate that typical elements have the form $[0, 0, \ldots, 0, x_1, x_2, \ldots, x_t]$.

(b) Prove Theorem 188 by studying the diagram

$$\begin{tikzcd}
\mathbb{R}\mathrm{P}^n \arrow[r, "\bar{\Delta}"] \arrow[d, equal] & \mathbb{R}\mathrm{P}^n \wedge \mathbb{R}\mathrm{P}^n \arrow[r, "x_s \wedge x_t"] \arrow[d] \arrow[rr, bend left, "x_s \bullet x_t"] & K_s \wedge K_t \arrow[r] & K_n \arrow[d, equal] \\
\mathbb{R}\mathrm{P}^n \arrow[r] \arrow[d, "j_n"'] & (\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{s-1}) \wedge (\mathbb{R}\mathrm{P}^n/\mathbb{R}\mathrm{P}^{t-1}) \arrow[r, "\bar{x}_s \wedge \bar{x}_t"] & K_s \wedge K_t \arrow[r] & K_n \\
S^n \arrow[r, "\cong"] & S^n. \arrow[u] \arrow[rru, "w"'] & &
\end{tikzcd}$$

Ring Structure in $H^*(\mathbb{C}\mathrm{P}^n)$ and $H^*(\mathbb{H}\mathrm{P}^n)$. The argument given in this section is easily adapted to the other projective spaces, with the added bonus that it is not necessary to impose any restrictions on the coefficients. We set

$$J_{\mathbb{C}} = \{x \in \mathbb{C} \mid |x| \leq 1\} \qquad \text{and} \qquad J_{\mathbb{H}} = \{x \in \mathbb{H} \mid |x| \leq 1\},$$

which are (or are homeomorphic to) D^2 and D^4, respectively. The constuctions of homotopies involving these disks should proceed exactly as before, leading to the following conclusion.

Theorem 189 *For any n, including ∞, and any commutative ring R with unit,*

(a) $H^*(\mathbb{C}\mathrm{P}^n; R) \cong R[x]/(x^{n+1})$, *where* $|x| = 2$.

(b) $H^*(\mathbb{H}\mathrm{P}^n; R) \cong R[x]/(x^{n+1})$, *where* $|x| = 4$.

If R is a ring with the property that $1 + 1 = 0$, then

(c) $H^*(\mathbb{R}\mathrm{P}^n; R) \cong R[x]/(x^{n+1})$, *where* $|x| = 1$.

PROBLEM 18.25 Prove Theorem 189.

18.5 Algebra Structures on $\mathbb{R}^n$ and $\mathbb{C}^n$

Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. If $\mathbb{F} \subseteq A$ is a normed $\mathbb{F}$-algebra, then we have a multiplication map

$$A \times A \to A.$$

with the following properties:

1. there is an identity element $e \in A$

2. μ respects length, in that $|x \cdot y| = |x||y|$ for any $x, y \in A$, and
3. μ commutes with scalars: $(cx) \cdot y = x \cdot (cy) = c(x \cdot y)$ for any $x, y \in A$ and $a \in \mathbb{F}$.

We know that we can define such multiplications on $\mathbb{R}$, $\mathbb{R}^2 = \mathbb{C}$ and $\mathbb{R}^4 = \mathbb{H}$. For what other values of n is there an algebra structure on $\mathbb{R}^n$?

Since these are contractible spaces, it would be understandable if you thought that homotopy theory could have no application to the study of such things. But the rigid algebraic structure allows us to derive strong conditions on the dimensions of A.

PROBLEM 18.26 Suppose $\mu : A \times A \to A$ is such a multiplication, and suppose that A is n-dimensional over $\mathbb{F}$.

(a) Show that μ restricts to a multiplication $\mu : S^{nd-1} \times S^{nd-1} \to S^{nd-1}$ such that the diagram

$$\begin{tikzcd}
S^{nd-1} \vee S^{nd-1} \arrow[d, "\mathrm{in}"'] \arrow[dr, "\mathrm{fold}"] & \\
S^{nd-1} \times S^{nd-1} \arrow[r, "\mu"'] & S^{nd-1}
\end{tikzcd}$$

commutes – i.e., S^{n-1} must be an H-space.

(b) Show that there is a multiplication $\mu : \mathbb{F}\mathrm{P}^{n-1} \times \mathbb{F}\mathrm{P}^{n-1} \to \mathbb{F}\mathrm{P}^{n-1}$ such that the diagram

$$\begin{tikzcd}
\mathbb{F}\mathrm{P}^{n-1} \vee \mathbb{F}\mathrm{P}^{n-1} \arrow[d, "\mathrm{in}"'] \arrow[dr, "\mathrm{fold}"] & \\
\mathbb{F}\mathrm{P}^{n-1} \times \mathbb{F}\mathrm{P}^{n-1} \arrow[r, "\mu"'] & \mathbb{F}\mathrm{P}^{n-1}
\end{tikzcd}$$

commutes – i.e., $\mathbb{F}\mathrm{P}^{n-1}$ must be an H-space.

(c) Apply what you know about the cohomology algebra $H^*(\mathbb{F}\mathrm{P}^{n-1})$ to obtain a diagram of graded rings and ring homomorphisms.

(d) By studying $\mu * (x_1)^n$, where x_1 generates $H^1(\mathbb{F}\mathrm{P}^{n-1})$, determine numerical conditions on n under which such a diagram exists. Conclude that there is no multiplication $\mu : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ unless n is ... what?

Your result implies the Fundamental Theorem of Algebra.[7]

Theorem 190 (Fundamental Theorem of Algebra) *The field $\mathbb{C}$ is algebraically closed.*

---

[7] Proof idea due to Bob Bruner.

Problem 18.27

(a) Show that if $\mathbb{C}$ is not algebraically closed, then there exists a finite-dimensional normed $\mathbb{C}$-algebra.

(b) Prove Theorem 190.

18.6 The Borsuk-Ulam Theorem

A map $f : S^m \to S^n$ is an odd map if $f(-x) = -x$ for all $x \in S^m$. We can use our understanding of $H^*(\mathbb{R}\mathrm{P}^n)$ to constrain the dimensions n and m for which an odd map can exist.

Theorem 191 *Let $f : S^m \to S^n$ be an odd map. Then $m \leq n$.*

Problem 18.28 Let $f : S^m \to S^n$ be an odd map.

(a) Show that the solid arrow part of the diagram

$$\begin{tikzcd} S^m \arrow[r, "f"] \arrow[d, "q_n"'] & S^n \arrow[d, "q_n"] \\ \mathbb{R}\mathrm{P}^m \arrow[r, "\phi"'] \arrow[ur, dotted, "g"] & \mathbb{R}\mathrm{P}^n \end{tikzcd}$$

commutes.

(b) Let $u_1 \in H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$ be the unique nonzero element, Show that if $\phi^*(u_1) = 0$, then there is a map g making the diagram commute.

(c) Show that $g \circ q_n = \pm f$.[8]

(d) Show that the formula $g \circ q_n = \pm f$ is incompatible with the oddness of f, and conclude that $\phi^*(u_1) \neq 0$.

(e) Prove Theorem 191.

Now we use Theorem 191 to prove the Borsuk-Ulam Theorem.

Theorem 192 *If $f : S^n \to \mathbb{R}^n$, then there is a point $x \in S^n$ such that $f(x) = f(-x)$.*

Problem 18.29 Prove Theorem 192 by studying the function $g : S^n \to S^{n-1}$ given by

$$g(x) = \frac{f(x) - f(-x)}{|f(x) - f(-x)|}.$$

---

[8] You may use the fact that q_n is a covering map, and hence lifts are uniquely determined by their value on a single point.

Think about what Theorem 192 says when $n = 2$. Given two maps $f, g : S^2 \to \mathbb{R}$, you can define $(f, g) : S^2 \to \mathbb{R}^2$, and so there must be at least one point $x \in S^2$ such that both $f(x) = f(-x)$ and $g(x) = g(-x)$. If we identify S^2 with the surface of the Earth, and let

$$f(x) = \text{temperature at } x \qquad \text{and} \qquad g(x) = \text{pressure at } x,$$

then we see that the Borsuk-Ulam theorem guarantees that, at any given time, there are two antipodal points on the Earth with exactly the same temperature and pressure.

The Ham Sandwich Theorem. Suppose you have a ham sandwich – that is, three sets, B_1, B_2 and $H \subseteq \mathbb{R}^3$, which represent the bread (top and bottom slice) and the ham, and you want to cut it neatly in half. That is, you want to divide each of the regions B_1, B_2 and H into two pieces of equal volume. The question is this: *can you cut the ham and the bread with a single knife slice?* More precisely, is there a single plane which divides each of B_1, B_2 and H into two pieces with equal volume?

The answer to this question is given by the **Ham sandwich theorem**.

Theorem 193 *Let $A_1, \ldots, A_n$ be (measurable) subsets of $\mathbb{R}^n$. Then there is an $(n-1)$-dimensional plane through the origin which bisects each of the sets A_i into two pieces of equal volume (measure).*

PROBLEM 18.30 For each $x \in S^n$, let $\pi_x \subseteq \mathbb{R}^n$ denote the $(n-1)$-dimensional subspace perpendicular to x. Then π_x divides $\mathbb{R}^n$ into two half-spaces: H_x^+, which contains x, and H_x^-, which contains $-x$. For each $i = 1, \ldots, n$, define a function $f_i : S^n \to \mathbb{R}$ by setting

$$f_i(x) = \text{measure of } A_i \cap H_x^+.$$

Then define $f = (f_1, \ldots, f_n) : S^n \to \mathbb{R}^n$. Prove Theorem 193 by applying the Borsuk-Ulam theorem to the function f.

Chapter 19

Uniqueness of Cohomology

Consider how we defined cohomology. First we constructed, with great effort, the sequence of spaces $K(G, n)$, and then we defined $\widetilde{H}^n(X; G) = [X, K(G, n)]$. Since $\Omega K(G, n+1) \simeq K(G, n)$, we were able to prove that cofiber sequences give rise to long exact sequences in cohomology, etc.

Now suppose I give you a cohomology theory $\widetilde{h}^*$, without telling you how it is constructed or defined. Wouldn't it be nice if there were spaces $L(n)$ and natural isomorphisms $\widetilde{h}^n(X) \cong [X, L(n)]$? That this is actually the case is the content of the famous **Brown Representability Theorem**. We use the Brown Representability theorem to show that for each abelian group G, there is a unique ordinary cohomology theory, on the category of CW complexes, with coefficients G.

We reprove this result by showing how to compute the ordinary cohomology of a CW complex using chain complexes.

Then we introduce **homology theories**, which are just like cohomology theories, but covariant. Many authors make the sloppy assertion that homology is dual to cohomology. In the sense that we have been using the term 'dual,' this is false. homology works well with domains, not targets. Thus we again have exact sequences from cofiber sequences, not fiber sequences. Thus, homology and cohomology cannot be dual in the sense of model categories. But there *is* an *algebraic* duality between them: we show how to view cohomology classes as functions defined on homology classes, and this leads to a map $H^*(X) \to \mathrm{Hom}(H_*(X), \mathbb{Z})$ from the cohomology of X to the 'dual of $H_*(X)$.'

We finish by establishing the Hurewicz theorem: if X is $(n-1)$-connected then $\pi_n(X) \cong H_n(X)$ (for $n \geq 2$).

19.1 Brown Representability and the Uniqueness of Ordinary Cohomology

We discussed representable functors in some detail in Chapter **??**. The Brown Representability Theorem gives conditions under which a homotopy functor $F : \mathcal{T}_* \to \textsc{ab}\,\mathcal{G}$ is representable.

Here are two properties that a homotopy functor $F : \mathcal{T}_* \to \textsc{ab}\,\mathcal{G}$ might satisfy:

1. *Wedge Axiom.* If $\{X_\alpha\}$ is any set of spaces, the natural comparison map $F(\bigvee X_\alpha) \to \prod F(X_\alpha)$ is an isomorphism.

2. *Mayer-Vietoris Axiom.* Apply F to a homotopy pushout square like so:

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & B \arrow[d, "j"] \\ C \arrow[r, "g"] & D \end{tikzcd} \qquad \begin{tikzcd} F(A) & F(B) \arrow[l, "i^*"'] \\ F(C) \arrow[u, "f^*"] & F(D). \arrow[l, "g^*"] \arrow[u, "j^*"'] \end{tikzcd}$$

For any pair of elements $u \in F(B)$ and $v \in F(C)$ such that $f^*(u) = i^*(v)$, there is an element $w \in F(D)$ such that $g^*(w) = v$ and $j^*(w) = u$. Pictorially:

$$\begin{tikzcd} t & u \arrow[l, mapsto] \\ v \arrow[u, mapsto] & w. \arrow[l, mapsto, dotted] \arrow[u, mapsto, dotted] \end{tikzcd}$$

These two properties are enough to guarantee representability of the functor F, at least for CW complexes.

thm:Brown

Theorem 194 (Brown Representability) *If $F : \textsc{h}\mathcal{T}_* \to \textsc{ab}\,\mathcal{G}$ is a contravariant homotopy functor that satisfies the Wedge and Mayer-Vietoris axioms, then*

(a) there is a CW complex E and an element $e \in F(E)$ such that the map

$$\Phi : [X, E] \to F(X) \quad \text{given by} \quad f \mapsto f^*(e)$$

is a natural isomorphism on the category of CW complexes, and

(b) the space E and the element $e \in F(E)$ are unique up to homotopy in the sense that if E' and $e' \in F(E)$ are another CW complex and element which represent F, then there is a homotopy equivalence, unique up to homotopy, $g : E \to E'$ such that $g^*(e') = e$.

The element $e \in F(E)$ is called a **fundamental class** for the functor F.

prob:Brown5Lemma

Problem 19.1 Let F be a homotopy functor satisfying the Wedge and Mayer-Vietoris properties.

(a) Show that if E is a space as in Theorem 194, then E is an abelian H-space.

(b) Show that if $A \to B \to C$ be a cofiber sequence, then the sequence

$$F(C) \leftarrow F(B) \leftarrow F(A) \leftarrow F(\Sigma C) \leftarrow F(\Sigma B)$$

is exact.

(c) Show that if E and $e \in F(E)$ are such that $\Phi : [S^n, E] \to F(S^n)$ is an isomorphism for all n, then Φ is an isomorphism for all CW complexes.

Problem 19.1 reduces the proof of Theorem 194 to showing that F can be represented for spheres. Our plan is to construct E skeleton by skeleton, with elements $e_n \in F(E_n)$ that represent F on spheres of dimension at most n. We start by setting $E_{-1} = *$ and $e_{-1} = 0 \in F(E_{-1})$.

Problem 19.2 Suppose we are given E_n and $e_n \in F(E_n)$ such that the transformation $\Phi : [S^k, E_n] \to F(S^k)$ given by $\Phi(f) = f^*(e_n)$ is an isomorphism for $k < n$ and surjective for $k = n$.

(a) Let $\ker(\Phi) = \{g_\beta \mid \beta \in \mathcal{J}\}$ and define $\widetilde{E}_{n+1}$ by the (homotopy) pushout square

$$\begin{tikzcd}
\bigvee_{\mathcal{J}} S^n \arrow[r, "(g_\beta)"] \arrow[d, "\mathrm{in}"'] \arrow[dr, phantom, "\mathrm{HPO}"] & E_n \arrow[d] \\
\bigvee_{\mathcal{J}} D^{n+1} \arrow[r] & \widetilde{E}_{n+1}.
\end{tikzcd}$$

Show that there is an element $\widetilde{e}_{n+1} \in F(\widetilde{E}_{n+1})$ such that $\Phi : [S^k, \widetilde{E}_{n+1}] \to F(S^k)$ is an isomorphism for all $k \leq n$.

(b) Construct the space E_{n+1} and the element $e_{n+1} \in F(E_{n+1})$ so that, in addition, $[S^{n+1}, E_{n+1}] \to F(S^{n+1})$ is surjective.

(c) Now we set $E = \operatorname{colim} E_n$, which is a CW complex by construction. Show that there is an element $e \in F(E)$ such that the inclusion $E_n \hookrightarrow E$ induces $e \mapsto e_n$.

Hint Express E as a (homotopy) pushout involving the spaces E_n.

(d) Prove Theorem 194(a).

The proof of Theorem 194(b) follows the typical pattern of proving uniqueness in category theory.

Problem 19.3 Suppose $e \in F(E)$ and $e' \in F(E')$ both represent F for all CW complexes X.

(a) Find maps $f : E \to E'$ and $g : E' \to E$ such that $g^*(e) = e'$ and $f^*(e') = e$.

(b) Prove Theorem 194(b) by showing that f and g are inverse homotopy equivalences.

The Wedge Axiom might be a little much to assume; what can we prove without it?

EXERCISE 19.4 Suppose $F(n)$ is a **finitely generated** abelian group for each n. Show that the functor F can be represented on the category of finite CW complexes. What can you say about the representing space? Is it unique?

Representation of Cohomology Theories. Theorem 194 applies to a very general class of functors, but we are primarily interested in the functors that constitute a cohomology theory.

prob:CohoRepr

PROBLEM 19.5 Let $\widetilde{h}^*$ be a cohomology theory satisfying the Wedge Axiom.

(a) Show that each functor $\widetilde{h}^n$ satisfies the Mayer-Vietoris property, and hence is representable by a space $E(n)$.

At least weakly equivalent.

(b) Show that $\Omega E(n+1) \simeq E(n)$.

HINT Use the Yoneda Lemma.

Problem 19.5 implies that the representing spaces $E(n)$ are infinite loop spaces.

An extremely important consequence of this theorem is that ordinary cohomology theories are completely determined by their coefficient groups, at least as long as you plug in CW complexes.

cor:OrdinaryUnique

Corollary 195 *Two reduced ordinary cohomology theories $\widetilde{H}^*$ and $\widetilde{J}^*$ with the same coefficient group G are naturally equivalent on the category $\mathcal{CW}$.*

PROBLEM 19.6 Prove Corollary 195.

HINT Describe the spaces that represent the homotopy functors $\widetilde{H}^n$ and $\widetilde{J}^n$.

EXERCISE 19.7 Criticize the following argument:

> If $\widetilde{h}^*$ and $\widetilde{k}^*$ have the same coefficient groups, then their representing spaces have the same homotopy groups, and hence are homotopy equivalent. Therefore all cohomology theories – ordinary or not – are determined by their coefficient groups.

EXERCISE 19.8 What can you say about cohomology theories which do not satisfy the Wedge Axiom?

19.2 Uniqueness via the Cellular Complex

Our goal here is to show how to use chain complexes to compute the cohomology of a CW complex. We will rederive the uniqueness of ordinary cohomology from this approach, but its real importance will become apparent only later, when certain algebraic topological constructions involving fibrations will result in recognizable chain complexes.

Throughout this section we will work with a reduced ordinary cohomology $\widetilde{H}^*$ with coefficients in an abelian group G. In particular, it could be that $\widetilde{H}^n(X) = [X, K(G,n)]$; but our argument will work equally well with any ordinary cohomology theory.

We will show that, no matter what $\widetilde{H}^*$ is, it can be calculated via chain complexes, at least for finite complexes. A CW complex X comes with a cone decomposition

$$\begin{tikzcd}
\bigvee S^0 \arrow[d] & \bigvee S^1 \arrow[d] & & \bigvee S^n \arrow[d] & \bigvee S^{n+1} \arrow[d] & \\
X_0 \arrow[r] & X_1 \arrow[r] & \cdots \arrow[r] & X_n \arrow[r] & X_{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

in which each 'L-shaped' sequence $\bigvee S^n \to X_n \to X^{n+1}$ is a cofiber sequence. Just as we did in our study of $\mathbb{R}\mathrm{P}^n$, we extend these sequences to obtain a diagram of long interwoven cofiber sequences

$$\begin{tikzcd}
 & \vdots \arrow[d] & & \vdots \arrow[d] & & \\
\bigvee S^{n-1} \arrow[r, dotted] & X_{n-1} \arrow[r] \arrow[d, dotted] & X_{n-1}/X_{n-2} \arrow[r] & \Sigma X_{n-2} \arrow[r] \arrow[d] & \Sigma(X_{n-2}/X_{n-3}) \arrow[r] & \cdots \\
\bigvee S^n \arrow[r] \arrow[rr, bend left, "\boxed{\delta_{n-1}}"] & X_n \arrow[r, dotted] \arrow[d] & X_n/X_{n-1} \arrow[r, dotted] \arrow[rr, bend right, "\boxed{\delta_n}"'] & \Sigma X_{n-1} \arrow[r] \arrow[d, dotted] & \Sigma(X_{n-1}/X_{n-2}) \arrow[r] & \cdots \\
\bigvee S^{n+1} \arrow[r] & X_{n+1} \arrow[r] \arrow[d] & X_{n+1}/X_n \arrow[r] & \Sigma X_n \arrow[r, dotted] \arrow[d] & \Sigma(X_n/X_{n-1}) \arrow[r, dotted] & \cdots \\
 & \vdots & & \vdots & &
\end{tikzcd}$$

The quotient X_n/X_{n-1} is a wedge of n-spheres, one for each n-cell of X, and $\Sigma(X_{n-1}/X_{n-2})$ is also a wedge of n-spheres, but this time there is one for each $(n-1)$-cell of X.

EXERCISE 19.9 Show that the big diagram above is functorial on the category of CW complexes and cellular maps.

Now X_n/X_{n-1} is a wedge of n-spheres, one for each n-cell of X. The quotients $q_i : X_n/X_{n-1} \to S_i^n$ to the i^{th} summand define an isomorophism

$$C^n(X) = \widetilde{H}^n(X_n/X_{n-1}) \cong \prod_{\substack{n\text{-cells} \\ \text{of } X}} G.$$

Then the maps δ_n define maps $d^n : C^n(X) \to C^{n+1}(X)$ given by the diagram

$$\begin{tikzcd}
C^{n+1}(X) \arrow[d, equal] & & C^n(X) \arrow[ll, "d^n"'] \arrow[d, equal] \\
\widetilde{H}^{n+1}(X_{n+1}/X_n) & \widetilde{H}^{n+1}(\Sigma(X_n/X_{n-1})) \arrow[l, "\delta_n^*"'] & \widetilde{H}^n(X_n/X_{n-1}) \arrow[l, "\cong"']
\end{tikzcd}$$

We write $\mathcal{C}^*(X)$ for the sequence of groups $C^n(X)$ and the homomorphisms d^n between them.

PROBLEM 19.10 Show that $d^{n+1} \circ d^n = 0$, so that the maps $d^n : C^n(X) \to C^{n+1}(X)$ make $\mathcal{C}^*(X)$ into a cochain complex.

EXERCISE 19.11

(a) Use two different cellular decompositions for S^1 to construct two different chain complexes for S^1. Then compute the cohomology groups of these chain complexes.

(b) Find a cellular homeomorphism from one of your decompositions to the other, and explain how it induces a chain map of chain complexes. What is the induced map on the (algebraic) cohomology of the chain complexes?

(c) Find a cellular map of degree 2, and determine the induced map on chain complexes, and then on the (algebraic) cohomology of the chain complexes.

We want to know that, for any space X, the cochain complex we have constructed depends only on the CW decomposition of the space X, and not on the choice of cohomology theory. As we have already shown, the *group* $C^n(X)$ depends only on X – specifically, the number of cells that X has in dimension n. It remains to study the coboundary map d^n.

Since d^n is essentially δ_n^*, we need to study the map δ_n^*; and since $\delta_n : X_{n+1}/X_n \to \Sigma(X_n/X_{n-1})$, it is a map from one wedge of spheres to another

$$\delta_n : \bigvee_{\substack{(n+1)\text{-cells} \\ \text{of } X}} S^{n+1} \to \bigvee_{\substack{n\text{-cells} \\ \text{of } X}} S^{n+1}.$$

Maps from one wedge of n-spheres to another are described by the integer matrix D_n which simply records the degrees of its coordinate functions.

prob:CellularNaturality

Problem 19.12

(a) We say that a CW complex has **finite type** if the given CW decomposition has only finitely many cells in each dimension. Show that if X is of finite type, then δ_n^* is determined by the (finite) matrix D_n^T, the transpose of D_n.

(b) Show that if $\widetilde{H}^*$ satisfies the Wedge Axiom, the finite type hypothesis is unnecessary.

(c) Show that if $f : X \to Y$ is a cellular map, then f induces a chain map $f^* : \mathcal{C}^*(Y) \to \mathcal{C}^*(X)$ and hence a map $f^* : \widetilde{H}^*(Y) \to \widetilde{H}^*(X)$. Conclude that $H^*(\mathcal{C}^*(\,?\,))$ is a functor on the category of CW complexes and cellular maps.

Here is our situation: let $\mathcal{CW}$ be the category of pointed finite CW complexes and cellular maps, let Chain be the category of chain complexes, and let $i : \mathcal{CW} \hookrightarrow \mathcal{T}$ be the inclusion functor. Then we have a diagram of categories and functors

$$\begin{tikzcd}
\mathcal{CW} \arrow[r, "i"] \arrow[d, "\mathcal{C}^*"'] & \mathcal{T}_* \arrow[d, "\widetilde{H}^*"] \\
\textsc{Chain} \arrow[r, "H^*"] & \textsc{ab}\,\mathcal{G}^*.
\end{tikzcd}$$

(where $\textsc{ab}\,\mathcal{G}^*$ is the category of graded abelian groups). We will show that – for any ordinary cohomology theory – the diagram commutes up to natural isomorphism.

Now let's prove that the algebraic homology of the cochain complex $\mathcal{C}^*(X)$ is actually isomorphic to the mysterious cohomology $\widetilde{H}^*(X)$.

thm:UniqueCohomology

Theorem 196 *For any reduced ordinary cohomology theory $\widetilde{H}^*$, there is a natural isomorphism $\Phi : \widetilde{H}^*(\,?\,) \to H^*(\mathcal{C}^*(\,?\,))$ defined on the category of finite CW complex and cellular maps. If $\widetilde{H}^*$ satisfies the Wedge Axiom then Φ is an isomorphism for all CW complexes.*

The proof involves a careful investigation of the following diagram, which is obtained by applying $\widetilde{H}^n$ to the large diagram of cofiber sequences that we constructed above.

$$\begin{tikzcd}
 & & & \widetilde{H}^n(X) \arrow[d, "\theta"] & \\
 & \widetilde{H}^{n-1}(X_n) \arrow[r] \arrow[d, dotted] & \widetilde{H}^n(X_{n+1}/X_n) \arrow[r] & \widetilde{H}^n(X_{n+1}) \arrow[d, "\phi"] & \\
\widetilde{H}^{n-1}(X_{n-1}/X_{n-2}) \arrow[r, "\delta"] \arrow[rr, bend left, "\boxed{d^{n-1}}"] & \widetilde{H}^{n-1}(X_{n-1}) \arrow[r, dotted, "\gamma"] \arrow[d] & \widetilde{H}^n(X_n/X_{n-1}) \arrow[r, dotted, "\beta"] \arrow[rr, bend right, "\boxed{d^n}"] & \widetilde{H}^n(X_n) \arrow[r, "\alpha"] \arrow[d, dotted] & \widetilde{H}^{n+1}(X_{n+1}/X_n) \\
 & \widetilde{H}^{n-1}(X_{n-2}) \arrow[r] & \widetilde{H}^{n-1}(X_{n-1}/X_{n-2}) \arrow[r] & \widetilde{H}^n(X_{n-1}). &
\end{tikzcd}$$

Before moving on to the meat of the proof, you should simplify this diagram as much as possible.

exer:Hom1

EXERCISE 19.13 Identify all groups in this diagram which are automatically zero. Which maps are necessarily onto? Which are one-to-one?

From our main diagram, let us separate out the following two maps:

$$C^n(X) = \widetilde{H}^n(X_n/X_{n-1}) \xrightarrow{\beta} \widetilde{H}^n(X_n) \xleftarrow{\phi\circ\theta} \widetilde{H}^n(X).$$

Let $Z_n = \ker(d^n)$ and $B_n = \operatorname{Im}(d^{n-1})$, so that

$$B_n \subseteq Z_n \subseteq \widetilde{H}^n(X_n/X_{n-1}) = C^n(X)$$

and $H^n(\mathcal{C}^*(X)) = Z_n/B_n$.

prob:Hom2

PROBLEM 19.14

(a) Show that $B_n = \ker(\beta)$. Conclude that $\widetilde{H}^n(\mathcal{C}^*(X)) \cong \beta(Z_n)$.

(b) Show that $\beta(Z_n) = \ker(\alpha)$.

(c) Show that if X is a finite CW complex, then $\ker(\alpha) \cong \widetilde{H}^n(X)$.

(d) Show that if $\widetilde{H}^*$ satisfies the cohomology wedge axiom (W^*) then $\ker(\alpha) \cong \widetilde{H}^n(X)$ for every CW complex X.

(e) Prove Theorem 196.

PROBLEM 19.15 Use Theorem 196 to reprove Corollary 195.

19.3 Homology Theories

In this section we introduce homology theories, which are very similar to cohomology theories, but covariant. Whether or not such functors exist is actually quite mysterious, because so far, every construction that has worked well with cofibrations, pushouts, etc. has been *contravariant.*

Definition 197 A (reduced) **homology theory** $\widetilde{h}_*$ is a sequence of covariant homotopy functors

$$\widetilde{h}_n : \mathcal{T}_* \to \text{AB}\,\mathcal{G}$$

such that

(a) there is a natural isomorphism $\widetilde{h}_n \circ \Sigma \to \widetilde{h}_{n-1}$.

(b) if $A \to B \to C$ is a cofiber sequence, then the sequence

$$\begin{tikzcd} \widetilde{h}_n(A) \arrow[r] & \widetilde{h}_n(B) \arrow[r] & \widetilde{h}_n(C) \end{tikzcd}$$

is exact.

To each reduced homology theory defined on $\mathcal{T}_*$, there is a corresponding **unreduced homology theory** h_* defined on $\mathcal{T}_\circ$ by the rule $h_*(X) = \widetilde{h}_*(X_+)$. The exactness property for the unreduced homology of a cofiber sequence is that

$$\begin{tikzcd} h_n(A) \arrow[r] & h_n(B) \arrow[r] & \widetilde{h}_n(C) \end{tikzcd}$$

should be exact for each n.

Problem 19.16 Show that if $\widetilde{h}_*$ is a homology theory, then each cofiber sequence gives rise naturally to a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{h}_n(A) \arrow[r] & \widetilde{h}_n(B) \arrow[r] & \widetilde{h}_n(C) \arrow[r] & \widetilde{h}_{n-1}(A) \arrow[r] & \cdots. \end{tikzcd}$$

Problem 19.17 Do the homotopy groups $\pi_n(\,?\,)$ define a homology theory?

Let's consider the homology of a wedge. The inclusions $\text{in}_j : X_j \hookrightarrow \bigvee_{\mathcal{J}} X_j$ induce maps $\widetilde{h}_*(X_j) \to \widetilde{h}_*(\bigvee_{\mathcal{J}} X_j)$ and hence a natural transformation

$$w : \bigoplus_{\mathcal{J}} \widetilde{h}_*(X_j) \to \widetilde{h}_*\left(\bigvee_{\mathcal{J}} X_j\right)$$

For finite wedges, w is an isomorphism, but just as for cohomology, a homology theory does not automatically behave well with respect to infinite wedges. The theory $\widetilde{h}_*$ satisfies the **homology Wedge Axiom** if w is an isomorphism for *every* wedge, infinite or finite.

The graded abelian group $\widetilde{h}_*(S^0)$ is called the **coefficient group** for the homology theory $\widetilde{h}$. If the coefficients are simply the abelian group G concentrated in dimension 0, then then $\widetilde{h}_*$ is called an **ordinary homology theory** with coefficients G.

or, maybe, the efficient group?

Now let X be a CW complex and let $\widetilde{H}_*$ be a reduced ordinary homology theory. Just as in the previous section, we can construct a chain complex $\mathcal{C}_*(X)$ by setting $C_n(X) = H_n(X_n/X_{n-1})$ with differential given by $(\delta_n)_*$.

The homology of X can be reconstructed from its chain complex.

Theorem 198 *Let $\widetilde{H}_*$ be a reduced ordinary homology theory, and let $\mathcal{C}_*(X)$ be the corresponding cellular chain complex. Then there is a natural isomorphism*

$$\Phi : \widetilde{H}_*(\,?\,) \longrightarrow H_*(\mathcal{C}_*(\,?\,)),$$

of functors defined on the category of finite CW complex and cellular maps. If $\widetilde{H}_*$ satisfies the homology Wedge Axiom then the functor $H_*(\mathcal{C}_*(\,?\,))$ can be extended to the category of all CW complexes, and Φ remains an isomorphism.

Corollary 199 *Any two homology theories with the same group of coefficients are naturally isomorphic on the category of finite CW complexes and cellular maps.*

This was quite an important theoretical result at the time it was discovered (the early 1940's). At that time, people were very excited about homology, and many topologists constructed their own homology theories to suit their particular purposes. In order to connect their work to that of other people, they needed to relate their homology to that of their friends. These proofs could be omitted once it was shown that there is really only one (ordinary) homology theory with coefficient group G.

PROBLEM 19.18 Prove Theorem 198 and Corollary 199

19.4 Some Examples of Homology Theories

We have plenty of experience with cohomology theories, but we have not yet seen a homology theory. Could it be that we have a definition without examples?

19.4.1 Stabilization of Maps

The Freudenthal Suspension Theorem implies that repeating the suspension operation can change a given homotopy group finitely many times.

PROBLEM 19.19 Show that for any space X the suspension maps $\Sigma : \pi_{n+t}(\Sigma^t X) \to \pi_{n+t+1}(\Sigma^{t+1} X)$ are isomorphisms for all sufficiently large t. How large is 'suffiently large?'

We say that the sequence of groups

$$\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \cdots \to \pi_{n+t}(\Sigma^t X) \to \pi_{n+t+1}(\Sigma^{t+1} X) \to \cdots$$

stabilizes for large t. This makes it easy to compute the colimit of the sequence: it is just the group that the sequence settles down on. It is called the n^{th} **stable homotopy group** of X. We denote it by $\pi_n^S(X)$, so that

$$\pi_n^S(X) = \operatorname{colim}\left(\cdots \to \pi_{n+t}(\Sigma^t X) \to \pi_{n+t+1}(\Sigma^{t+1} X) \to \cdots\right).$$

There is a canonical map $\Sigma^\infty : \pi_n(X) \to \pi_n^S(X)$ which just takes α to the element represented by $\Sigma^t\alpha$, where t is large enough, or larger.

It will be convenient to establish some additional notation for suspension. Suppose we suspend t times, and then do the rest of the infinite suspension, like so:

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\Sigma^t"'] \arrow[rr, bend left, "\Sigma^\infty"] & \pi_{n+t}(\Sigma^t X) \arrow[r, "\boxed{\Sigma^{\infty-t}}"'] & \pi_n^S(X).
\end{tikzcd}$$

We call the boxed map $\Sigma^{\infty-t}$ instead of Σ^∞ because, by our definition, the target of $\Sigma^\infty : \pi_{n+t}(\Sigma^t X) \to \pi_{n+t}^S(\Sigma^t X)$, is $\pi_{n+t}^S(\Sigma^t X)$, not $\pi_n^S(X)$.

Theorem 200 *The stable homotopy groups $\pi_*^S(\,?\,)$ define a homology theory that satisfies the homology wedge axiom.*

Since π_n^S is clearly a homotopy functor, you can prove the theorem by checking that π_*^S satisfies the two defining properties of a homology theory.

PROBLEM 19.20

(a) Construct a natural isomorphism $s_X : \pi_{n+1}^S(\Sigma X) \to \pi_n^S(X)$.

(b) Show that, if $A \to B \to C$ is a cofiber sequence, then

$$\begin{tikzcd}
\pi_n^S(A) \arrow[r] & \pi_n^S(B) \arrow[r] & \pi_n^S(C)
\end{tikzcd}$$

is exact.

HINT Since you are trying to get an exact sequence using a cofiber sequence in the target, you should use Theorem **??**.

Stable homotopy groups were originally introduced as a 'first approximation' to ordinary homotopy groups. Since then, the idea of 'working stably,' i.e., assuming that all spaces have been suspended so much that further suspension has no effect besides increasing indices, has taken on a life of its own. Let's look at one more example.

Problem 19.21 Show that for any finite-dimensional CW complex A, the functors

$$\widetilde{h}_n(X) = \operatorname{colim}\big(\cdots \to [\Sigma^t(\Sigma^n A), \Sigma^t X] \to [\Sigma^{t+1}(\Sigma^n A), \Sigma^{t+1} X] \to \cdots\big)$$

constitute a homology theory. Notice that it makes perfect sense to plug in **negative** values for n!

19.4.2 Ordinary Homology

Stable homotopy groups are very nice, but they are also very complicated. It is known that the coefficient groups $\pi_*^S(S^0)$ are nonzero for infinitely many values of $*$, so π_*^S is very far from being an ordinary homology theory.

Fix an abelian coefficient group G. For each t, let $\lambda_t : \Sigma K(G,t) \to K(G,t{+}1)$ be the adjoint of the identity map $K(G,t) \to K(G,t) \simeq \Omega K(G,t{+}1)$. Now we define $\tau_t = (\mathrm{id}_X \wedge \lambda_t)_* \circ \Sigma$, as in the diagram

$$\begin{tikzcd}
\pi_{n+t}(X \wedge K(G,t)) \arrow[r, "\Sigma"] \arrow[dr, "\tau_t"'] & \pi_{n+t+1}(X \wedge \Sigma K(G,t)) \arrow[d, "(\mathrm{id}_X \wedge \lambda_t)_*"] \\
& \pi_{n+t+1}(X \wedge K(G,t+1)).
\end{tikzcd}$$

Problem 19.22

(a) Show that the map $\Sigma K(G,t) \to K(G,t+1)$ is a $(2t-1)$-equivalence.

Hint What are the homotopy groups of $K(G,t+1)$? What can you say about the homotopy groups of $\Sigma K(G,t)$?

(b) Show that, for fixed n and large enough t, the composite map

$$\tau_t : \pi_{n+t}(X \wedge K(G,t)) \to \pi_{n+t+1}(X \wedge K(G,t+1))$$

is an isomorphism.

Hint The first map, Σ, can be analyzed using the Freudenthal Suspension theorem; for the second one, what can you say about the cofiber of $\mathrm{id}_X \wedge \lambda_t$?

It follows from this exercise that the sequence of groups eventually becomes constant. We'll use the notation

$$\widetilde{H}_n(X;G) = \operatorname{colim}\Big(\cdots \to \pi_{n+t}(X \wedge K(G,t)) \xrightarrow{\tau_t} \pi_{n+t+1}(X \wedge K(G,t+1)) \to \cdots\Big)$$

to denote the limit group. This is called the **homology** of X with coefficients in the group G. It remains to show that the functors $\widetilde{H}^*$ actually do define a homology theory.

Problem 19.23

(a) Show that $\widetilde{H}^n(\,?\,;G)$ is a functor.

(b) Construct a natural isomorphism $s : \widetilde{H}_n(\Sigma X;G) \to \widetilde{H}_{n-1}(X;G)$.

(c) Show that, if $A \to B \to C$ is a cofiber sequence, then

$$\widetilde{H}_n(A;G) \longrightarrow \widetilde{H}_n(B;G) \longrightarrow \widetilde{H}_n(C;G)$$

is exact.

Hint Use the fact that $A \wedge K \to B \wedge K \to C \wedge K$ is a cofiber sequence and Problem **??**.

(d) Determine $\widetilde{H}_k(S^n;G)$ for all k.

You have proved the following.

Theorem 201 *The functors $\widetilde{H}_*$ define a reduced ordinary homology theory with coefficients in the group G.*

Homology Generators. We end this section by choosing a particular generator for the group $\widetilde{H}_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$. We constructed $K(\mathbb{Z},n)$ by killing all the higher homotopy groups in the space S^n; thus $K(\mathbb{Z},n)$ comes with a map $i_n : S^n \to K(\mathbb{Z},n)$ which induces an isomorphism

$$(i_n)_* : \pi_n(S^n) \to \pi_n(K(\mathbb{Z},n)) = \widetilde{H}^n(S^n;\mathbb{Z}).$$

The group $\pi_n(S^n)$ is generated by $\mathrm{id}_{S^n} : S^n \to S^n$, and so $\pi_n(K(\mathbb{Z},n))$ is generated by $(i_n)_*(\mathrm{id}) = i_n$. This is our preferred generator for $H^n(S^n;\mathbb{Z})$. For homology, we have

$$\pi_{n+t}(S^n \wedge K(\mathbb{Z},t)) \cong \mathbb{Z} \quad \text{is generated by} \quad \mathrm{id} \wedge i_t,$$

which is our preferred generator; we will call it $s_n \in H_n(S^n)$.

Problem 19.24 Show that the composite isomorphism

$$\Sigma : H_n(S^n;\mathbb{Z}) \xrightarrow{\cong} H_{n+1}(\Sigma S^n;\mathbb{Z}) \xrightarrow{\cong} H_{n+1}(S^{n+1};\mathbb{Z})$$

carries s_n to s_{n+1}.

19.5 Generalization

We have shown that any cohomology theory is represented by a sequence of spaces $E(n)$ together with homotopy equivalences $E(n) \to \Omega E(n+1)$. In our construction of ordinary homology, we used the representing spaces for ordinary cohomology (the Eilenberg-Mac Lane spaces), but the maps we used were the adjoints $\Sigma E(n) \to E(n+1)$, which are not homotopy equivalences. In the late 1950's G.W. Whitehead and Lima made the big conceptual leap: the homotopy equivalence is immaterial to the construction!

If we omit the condition of homotopy equivalence, we are led to the concept of a *spectrum.* A **spectrum** is a collection $\mathbf{E} = \{E(n), \epsilon_n\}$, where $\epsilon_n : \Sigma E(n) \to E(n+1)$ is an arbitrary map.

Given a spectrum $\mathbf{E}$, we can define, for each space X, maps

$$\begin{tikzcd}
{[\Sigma^t X, E(n+t)]} \arrow[rr] \arrow[dr, "\Sigma"'] & & {[\Sigma^{t+1} X, E(n+t+1)]} \\
& {[\Sigma^{t+1} X, \Sigma E(n+t)]} \arrow[ur, "(\epsilon_{n+t})_*"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
{[S^{n+t}, E(t) \wedge X]} \arrow[rr] \arrow[dr, "\Sigma"'] & & {[\Sigma^{n+t+1} X, E(t+1)]} \\
& {[S^{n+t+1} X, \Sigma E(t) \wedge X]} \arrow[ur, "(\epsilon_t \wedge \mathrm{id}_X)_*"'] &
\end{tikzcd}$$

Then we may define

$$\begin{aligned}
\widetilde{\mathbf{E}}^n(X) &= \operatorname{colim}\left(\cdots \to [\Sigma^t X, E(n+t)] \to [\Sigma^{t+1} X, \Sigma E(n+t)] \to \cdots\right), \qquad \text{and} \\
\widetilde{\mathbf{E}}_n(X) &= \operatorname{colim}\left(\cdots \to [S^{n+t}, E(t) \wedge X] \to [S^{n+t+1} X, \Sigma E(t) \wedge X] \to \cdots\right).
\end{aligned}$$

The second formula is reminiscent of our definition for ordinary homology.

prob:SpectralHomology1

PROBLEM 19.25 Let $\mathbf{E}$ be a spectrum.

(a) Show that $\widetilde{\mathbf{E}}_*$ and $\widetilde{\mathbf{E}}^*$ are homotopy functors.

(b) Construct natural isomorphisms $\widetilde{\mathbf{E}}_{n+1} \circ \Sigma \to \widetilde{\mathbf{E}}_n$ and $\widetilde{\mathbf{E}}^{n+1} \circ \Sigma \to \widetilde{\mathbf{E}}^n$.

thm:SpectralHomology

Theorem 202 *The functors $\widetilde{\mathbf{E}}_n$ and the isomorphisms of Problem* **??** *constitute a homology theory, and similarly $\widetilde{\mathbf{E}}^*$ is a cohomology theory.*

PROBLEM 19.26 Prove Theorem 202 by showing that a cofiber sequence $A \to B \to C$ gives rise to a exact sequences

$$\widetilde{\mathbf{E}}_n(A) \to \widetilde{\mathbf{E}}_n(B) \to \widetilde{\mathbf{E}}_n(C) \qquad \text{and} \qquad \widetilde{\mathbf{E}}^n(A) \leftarrow \widetilde{\mathbf{E}}^n(B) \leftarrow \widetilde{\mathbf{E}}^n(C).$$

HINT A colimit of exact sequences of abelian groups is again exact.

If we are given a cohomology theory $\widetilde{h}^*$, we can apply the Brown Representability Theorem to obtain a spectrum**E** that represents $\widetilde{h}^*$ in the sense that $\widetilde{h}^n(\,?\,) \cong [\,?\,, E(n)]$. Now we have a new cohomology theory related to **E**, namely $\widetilde{\mathbf{E}}^*$.

PROBLEM 19.27 Show that the theories $\widetilde{h}^*$ and $\widetilde{\mathbf{E}}^*$ are naturally equivalent on the category of finite CW complexes.

There is a version of the Brown Representability theorem for homology, which says that every homology theory can be constructed from a spectrum as above. Thus we have correpsondences

$$\{\text{cohomology theories}\} \longleftrightarrow \{\text{spectra}\} \longleftrightarrow \{\text{homology theories}\}.$$

EXERCISE 19.28 Show that homology theories are in bijective correspondence with cohomology theories (defined on the category of finite CW complexes).

19.6 Relating Homology to Cohomology

We have seen that each homology theory $\widetilde{h}_*$ gives rise to a unique corresponding cohomology theory $\widetilde{h}^*$, and vice versa. This suggests that it might be possible to compute $\widetilde{h}^*(X)$ from $\widetilde{h}_*$ (or the reverse). We show how to treat ordinary cohomology classes as functions defined on ordinary homology. We then state, and sketch a proof of, a universal coefficient theorem relating cohomology to homology. This is the result underlying the algebraic duality between homology and cohomology.

Let $u \in \widetilde{H}^k(X;A)$ and $\alpha \in \widetilde{H}_n(X;B)$. Thus u is a homotopy class of maps $X \to K(A,k)$ and α is represented by a map

$$\alpha : S^{n+t} \to X \wedge K(B,t)$$

for sufficiently large t. We define $u(\alpha)$ to be the class represented by the composition

$$\begin{tikzcd}
S^{n+t} \arrow[r, "\alpha"] \arrow[ddd, bend right, "u(\alpha)"'] & X \wedge K(B,t) \arrow[d, "\Delta_X \wedge \mathrm{id}"] \\
 & (X \wedge X) \wedge K(B,t) \arrow[d, "\mathrm{id} \wedge u \wedge \mathrm{id}"] \\
 & X \wedge (K(A,k) \wedge K(B,t)) \arrow[d, "\mathrm{id} \wedge \mu"] \\
 & X \wedge K(A \otimes B, k+t).
\end{tikzcd}$$

Notice that $u(\alpha) \in \widetilde{H}_{n-k}(X; A \otimes B)$, so the cohomology class $u \in \widetilde{H}^k(X; A)$ defines a function

$$u : \widetilde{H}_n(X; B) \to H_{n-k}(X; A \otimes B).$$

The rule $(u, \alpha) \mapsto u(\alpha)$ is usually denoted

$$(u, \alpha) \mapsto \langle u, \alpha \rangle$$

(especially in the case $n = k$), and it defines a function

$$\widetilde{H}^k(X; A) \times \widetilde{H}_n(X; B) \to \widetilde{H}_{n-k}(X; A \otimes B).$$

We want to understand this function.

Problem 19.29 Show that $\langle\, ?\, ,\, ?\, \rangle$ is natural in both variables. That is, suppose $f : X \to Y$, $u \in \widetilde{H}^*(Y)$, $\alpha \in \widetilde{H}_*(X)$. Then we can form

$$\langle u, f_*(\alpha) \rangle \in H_{n-k}(Y) \quad \text{and} \quad \langle f^*(u), \alpha \rangle \in H_{n-k}(X).$$

Show that $f_*(\langle f^*(u), \alpha \rangle) = \langle u, f_*(\alpha) \rangle$.

Problem 19.30 Show that ϕ is bilinear, so that it induces a map

$$\langle\, ?\, ,\, ?\, \rangle : \widetilde{H}^k(X; A) \otimes \widetilde{H}_n(X; B) \to \widetilde{H}_{n-k}(X; A \otimes B).$$

Clearly we can plug in X_+ for X, and so we get a map

$$\langle\, ?\, ,\, ?\, \rangle : H^n(X; A) \otimes H_n(X; B) \to H_{n-k}(X; A \otimes B).$$

defined and natural for unpointed spaces.

The Case $n = k$. We work now in the unpointed context, and specialize to the case $n = k$ (and X is path-connected). In this case our map has the form

$$\langle\, ?\, ,\, ?\, \rangle : H^n(X; R) \otimes H_n(X; R) \to H_0(X; R \otimes R) \cong R.$$

Furthermore, if $f : X \to Y$, then the induced map on H_0 is simply the identity on R, so that naturality reduces to

$$\langle f^*(u), \alpha \rangle = \langle u, f_*(\alpha) \rangle.$$

Thus, the rule $u \mapsto \langle u,\, ?\, \rangle$ defines a homomorphism

$$u : H^n(X; R) \to \mathrm{Hom}_R(H_n(X; R), R).$$

Theorem 203 *If R is a field, then the natural transformation*

$$u : H^n(X;R) \to \operatorname{Hom}_R(H_n(X;R), R)$$

is an isomorphism for all finite CW complexes X.

PROBLEM 19.31 Suppose R is a field.

(a) Show that $\widetilde{h}^n(X) = \operatorname{Hom}_R(H_n(X), R)$ is a cohomology theory defined on the category of finite CW complexes.

(b) Show that u is a natural transformation of homology theories.

(c) Prove Theorem 203.

PROBLEM 19.32 Can you weaken the hypotheses in Theorem 203? What if R is an ordinary ring but you know $H_n(X;R)$ is a free R-module?

19.7 The Hurewicz Theorem

We finish this chapter with an account of the celebrated Hurewicz theorem, which relates ordinary homology to homotopy groups.

We want to relate $\pi_*(X)$ and $\widetilde{H}_*(X;\mathbb{Z})$, which means finding a map joining them. If $\alpha \in \pi_n(X)$ then $\alpha : S^n \to X$, and so it induces a map

$$\alpha_* : \widetilde{H}_n(S^n;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}).$$

If we apply this induced map to the canonical generator $s_n \in \widetilde{H}_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$, we get an element $\alpha_*(s_n) \in \widetilde{H}_n(X;\mathbb{Z})$. The function

$$\mathcal{H} : \pi_n(X) \to \widetilde{H}_n(X;\mathbb{Z}) \quad \text{given by} \quad \mathcal{H}(\alpha) = \alpha_*(s_n)$$

is called the **Hurewicz map**.

We begin by observing that the Hurewicz map is essentially a stable gadget.

PROBLEM 19.33

(a) Show that the diagram

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\Sigma"] \arrow[d, "\mathcal{H}"'] & \pi_{n+1}(\Sigma X) \arrow[d, "\mathcal{H}"] \\
\widetilde{H}_n(X;\mathbb{Z}) \arrow[r, "\Sigma", "\cong"'] & \widetilde{H}_{n+1}(\Sigma X;\mathbb{Z})
\end{tikzcd}$$

is commutative.

(b) Show that there is a map $\mathcal{H}^S : \pi_n^S(X) \to \widetilde{H}_n(X;\mathbb{Z})$ which makes the diagram

$$\begin{tikzcd}
\pi_n(X) \arrow[r, "\Sigma^\infty"] \arrow[dr, "\mathcal{H}"'] & \pi_n^S(X) \arrow[d, "\mathcal{H}^S"] \\
& \widetilde{H}_n(X;\mathbb{Z})
\end{tikzcd}$$

commute.

The map $\mathcal{H}^S$ is called the **stable Hurewicz map**.

PROBLEM 19.34 Let $\alpha \in \pi_n^S(X)$, and let t be large enough that there are isomorphisms

$$\Sigma^{\infty-t} : \pi_{n+t}(X \wedge S^t) \xrightarrow{\cong} \pi_n^S(X)$$

and

$$\Sigma^{\infty-t} : \pi_{n+t}(X \wedge K(\mathbb{Z},t)) \xrightarrow{\cong} \widetilde{H}_n(X;\mathbb{Z}).$$

Let $i_t : S^t \to K(\mathbb{Z},t)$ be the inclusion of the bottom cell, as discussed above. Show that the Hurewicz map can be identified with $(\mathrm{id}_X \wedge i_t)_*$ in the sense that the diagram

$$\begin{tikzcd}
\pi_{n+t}(X \wedge S^t) \arrow[r, "\Sigma^{\infty-t}", "\cong"'] \arrow[d, "(\mathrm{id}_X \wedge i_t)_*"'] & \pi_n^S(X) \arrow[d, "\mathcal{H}^S"] \\
\pi_{n+t}(X \wedge K(\mathbb{Z},t)) \arrow[r, "\Sigma^{\infty-t}", "\cong"'] & \widetilde{H}_n(X;\mathbb{Z})
\end{tikzcd}$$

commutes.

Problem 19.34 immediately implies the following basic result.

Proposition 204 *The Hurewicz map $\mathcal{H} : \pi_n(X) \to \widetilde{H}_n(X;\mathbb{Z})$ and the stable Hurewicz map $\mathcal{H}^S : \pi_n^S(X) \to \widetilde{H}_n(X;\mathbb{Z})$ define natural transformations of functors $\mathcal{T}_* \to \mathcal{G}$.*

EXERCISE 19.35 Prove it.

Now we come to the **Hurewicz theorem**.

Theorem 205 *Let X, be an $(n-1)$-connected space. Then*

(a) For any n,

  i. The stable Hurewicz map $\mathcal{H}^S : \pi_n^S(X) \to \widetilde{H}_n(X;\mathbb{Z})$ is an isomorphism, and

  ii. $\mathcal{H}^S : \pi_{n+1}^S(X) \to H_{n+1}(X;\mathbb{Z})$ is onto.

(b) If $n > 1$ then

i. $\mathcal{H} : \pi_n(X) \to H_n(X; \mathbb{Z})$ *is an isomorphism, and*

ii. $\mathcal{H} : \pi_{n+1}(X) \to H_{n+1}(X; \mathbb{Z})$ *is onto.*

iii. The Hurewicz map $\pi_1(X) \to H_1(X; \mathbb{Z})$ *is abelianization.*

PROBLEM 19.36 Prove Theorem 205 as follows.

(a) Let $f : A \to B$ be a $(t+1)$-equivalence between simply-connected spaces. How much of an equivalence is $\mathrm{id}_X \wedge f : X \wedge A \to X \wedge B$?

(b) Show that, in the notation of Problem 19.34, the map $\mathrm{id}_X \wedge i_t$ is a $(t+1)$-equivalence.

(c) Prove Theorem 205 for $n > 1$

(d) Prove the $n = 1$ case by studying the diagram

$$\begin{tikzcd}
\pi_1(X) \arrow[r] \arrow[d, "\Sigma"'] & H_1(X) \arrow[d, "\Sigma"] \\
\pi_2(\Sigma X) \arrow[r] & H_2(\Sigma X).
\end{tikzcd}$$

Chapter 20

Applications of Cohomology Rings

20.1 Computations

In this section you will strengthen your cohomological muscles by computing some examples.

PROBLEM 20.1 Compute the cohomology ring of $\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1}$, using as coefficients $\mathbb{Z}$ and $\mathbb{Z}/n$ for $n \in \mathbb{N}$.

Surfaces. A surface is a 2-dimensional manifold. There is a complete classification of the compact surfaces, in terms of an operation called the **connected sum**. If M and N are 2-dimensional manifolds, then we may cut small disks out of each, leaving compact manifolds $\overline{M}$ and $\overline{N}$, each having boundary S^1. Now form the (homotopy) pushout diagram

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & \overline{M} \arrow[d] \\
\overline{N} \arrow[r] & M\#N.
\end{tikzcd}$$

The space $M\#N$ is a compact 2-manifold without boundary, and it is called the **connected sum** of M and N. The classification of compact surfaces is

> Every orientable surface is a connected sum of finitely many tori $S^1 \times S^1$, and every nonorientable surface is a connected sum of an orientable surface and $\mathbb{R}\mathrm{P}^2$.

You should take this result for granted for the purposes of these problems. Note that the classification, as stated, does not assert that the manifolds on this list are pairwise nonhomeomorphic to each other.

PROBLEM 20.2 Let $T = S^1 \times S^1$ be the torus, and let $\overline{T}$ be the space obtained by deleting the interior of a small disk from T. Let $j : S^1 \to \overline{T}$ be the inclusion of the boundary of the disk.

(a) Show that j is not nullhomotopic, but its suspension is trivial.[1]

(b) Show that the induced maps $j_* : H_*(S^1) \to H_*(\overline{T})$ and $j^* : H^*(\overline{T}) \to H^*(S^1)$ are trivial for all coefficients.

PROBLEM 20.3 Determine the cohomology ring $H^*(M; \mathbb{Z})$, where M is an orientable compact 2-manifold. Show that if M and N are connected sums of different numbers of tori, then they are not homotopy equivalent, so there are no repetitions in the list of orientable 2-manifolds.

PROBLEM 20.4 Determine the cohomology ring $H^*(M; \mathbb{Z}/2)$, where M is a nonorientable compact 2-manifold. Show that if M and N are connected sums of different numbers of tori, then they are not homotopy equivalent, so there are no repetitions in the list of nonorientable 2-manifolds.

PROBLEM 20.5 The Klein bottle K is one of the spaces on the list; which one?

Euler Characteristic. Let X be a finite-dimensional space such that $H_n(X; \mathbb{Z})$ is finitely generated for each n. Then we define the **Euler characteristic** of X to be

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \dim_{\mathbb{F}}(H_k(X; \mathbb{Z}/p))$$

where p is any prime.

PROBLEM 20.6 Show that surfaces are classified by their Euler characteristic.

PROBLEM 20.7 What is the Euler characteristic of a Moore space?

It is a bit funny that we have not attempted to assert any control over the prime p used to compute the Euler characteristic. To see that the choice is immaterial, we need to take a brief algebraic detour, and define the Euler characteristic of a free graded R-module A_* by setting

$$\chi(A_*) = \sum_{k=0}^{\infty} (-1)^k \dim_R(A_k).$$

---

[1] You may take for granted that, because of the homogeneity of a manifold, if j_1 and j_2 are obtained by the deletion of different disks, then they are homotopy equivalent to one another.

Problem 20.8 Show that if C_* is a chain complex, then

$$\chi(C_*) = \chi(H_*(C_*)).$$

Problem 20.9 Show that $\chi(X)$ is independent of the choice of prime.

Problem 20.10 Show that if X is a finite CW complex, then

$$\chi(X) = \sum_{k=0}^{\infty} (-1)^k \, (\text{number of } k\text{-cells in } X)\,.$$

Problem 20.11 Show that $\chi(X \times Y) = \chi(X)\chi(Y)$.

20.2 Hopf Invariants of Maps $S^{2n-1} \to S^n$

The Hopf map $p : S^3 \to S^2$ was first shown to be nontrivial without using the long exact sequence of a fibration. What did Hopf do? He noticed that for each $x \in S^2$, the preimage $p^{-1}(x)$ is a circle; and furthermore, if $x \neq y \in S^2$, then the circles $p^{-1}(x)$ and $p^{-1}(y)$ are *linked*. So Hopf defined a function on maps like this: let $f : S^3 \to S^2$;

(a) If f is not differentiable, then replace f with a differentiable map which is homotopic to f;

(b) Then almost every point $x \in S^2$ will have a circle for a preimage, take any two such ‘nice’ points, and see how many times their preimages are linked – this number is called the **Hopf invariant** of f, and it is denoted $h(f)$.

He proved that $h(f)$ is a well-defined *homomorphism* $\pi_3(S^2) \to \mathbb{Z}$, so that if $f \simeq g$, then $h(f) = h(g)$ and, of course, $h(*) = 0$. Then since the Hopf map has $h(p) = 1$, it must be that $p \not\simeq *$.

About 15 years later, Steenrod recast the Hopf invariant in terms of cohomology. Let $f : S^{2n-1} \to S^n$, and let C_f be the cofiber of f. Then $\widetilde{H}^k(C_f;\mathbb{Z})$ is nontrivial only in dimensions n and $2n$; assign generators x_n and y_{2n} to these groups. Then

$$x_n^2 = a \cdot y_{2n}$$

for some integer $a \in \mathbb{Z}$; Steenrod showed that if the generator y is chosen correctly, $a = h(f)$. We take this cup product formula as our definition of the Hopf invariant.

The Hopf Invariant is a Homomorphism. We will follow an argument due to James and prove that h is a homomorphism.

Proposition 206 *The Hopf invariant is a homomorphism* $\pi_{2n-1}(S^n) \to \mathbb{Z}$.

It will be convenient to use a nonstandard formulation of the condition that a map be a homomorphism.

Problem 20.12 Show that the following are equivalent.

1. $h : \pi_{2n-1}(S^n) \to \mathbb{Z}$ is a homomorphism.
2. if $\alpha_1 + \alpha_2 + \alpha_3 = 0 \in \pi_{2n-1}(S^n)$, then $h(\alpha_1) + h(\alpha_2) + h(\alpha_3) = 0 \in \mathbb{Z}$.

You will verify condition (b). So suppose $\alpha_1 + \alpha_2 + \alpha_3 = 0 \in \pi_{2n-1}(S^n)$. Then we form three spaces,

$$X_1 = S^n \cup_{\alpha_1} D^{2n}, \qquad X_2 = S^n \cup_{\alpha_2} D^{2n} \qquad \text{and} \qquad X_3 = S^n \cup_{\alpha_3} D^{2n}.$$

We also study the space Y obtained by attaching three disks to S^n, by the maps α_1, α_2 and α_3. There are natural inclusions $X_i \hookrightarrow Y$ for $i = 1, 2, 3$.

Let's write $a_1 = h(\alpha_1)$, $a_2 = h(\alpha_2$ and $a_3 = h(\alpha_3)$.

Problem 20.13

(a) Determine the cohomology rings $H^*(X_i; \mathbb{Z})$ for $i = 1, 2, 3$.

(b) Determine the cohomology ring $H^*(Y; \mathbb{Z})$ and the induced maps $H^*(Y; \mathbb{Z}) \to H^*(X_i; \mathbb{Z})$ for $i = 1, 2, 3$.

(c) Consider the cofiber sequence

$$\bigvee_1^3 S^{2n-1} \xrightarrow{(\alpha_1,\alpha_2,\alpha_3)} S^n \longrightarrow X \xrightarrow{q} \bigvee_1^3 S^{2n},$$

and show that there is a map $f : S^{2n} \to X$ such that $q_*(f)$ is the three-fold folding map $\nabla_3 \in \pi_{2n}(\bigvee_1^3 S^2 n)$.

(d) Prove Proposition 206 by studying the induced map $f^* : H^*(Y; \mathbb{Z}) \to H^*(S^{2n}; \mathbb{Z})$.

Basic Computations of Hopf Invariants. Let's do some computations.

Problem 20.14

(a) Show that if n is odd, the Hopf invariant of every map $f : S^{2n-1} \to S^n$ is zero.

(b) Using what you know about the cup products in $H^*(J_2(S^{n+1}))$ show that if n is even, then there is a map $f : S^{2n-1} \to S^n$ with $h(f) = 2$.

Your work shows that if n is even, then the image of the Hopf invariant is either all of $\mathbb{Z}$ or else $2\mathbb{Z} \subseteq \mathbb{Z}$. It will be all of $\mathbb{Z}$ if and only if there is an element $f \in \pi_{2n-1}(S^n)$ with Hopf invariant one.

Because $\mathbb{C}\mathrm{P}^1 \cong S^2$, the canonical quotient map $S^3 \to \mathbb{C}\mathrm{P}^1$ defines an element $\eta \in \pi_3(S^2)$.

Problem 20.15

(a) Show that $\pi_3(S^2) \cong \mathbb{Z} \cdot \eta$.

(b) Show that $h : \pi_3(S^2) \to \mathbb{Z}$ is an isomorphism.

Problem 20.16 Similarly $\mathbb{H}\mathrm{P}^1 \cong S^4$ and so the quotient map defines an element $\nu \in \pi_7(S^4)$. Determine $h(\nu)$. Is $\pi_7(S^4) \cong \mathbb{Z} \cdot \nu$?

Determining the complete list of values of n for which there a map with Hopf invariant equal to one was a major problem throughout the 1950's. It was finally solved by J. F. Adams in a monumental 100-page paper. There only maps $f \in \pi_{2n-1}(S^n)$ with Hopf invariant one for $n = 1, 2, 4$ or 8, and no other values of n.

The Group $\pi_{n+1}(S^n)$. We can use our knowledge of $\pi_3(S^2)$ together with our understanding of the suspension homomorphism to determine $\pi_{n+1}(S^n)$ for all n.

Problem 20.17

(a) Determine the kernel of $\Sigma : \pi_3(S^2) \to \pi_4(S^3)$.

Hint Study the cellular structure of $J(S^2)$.

(b) Determine the group $\pi_{n+1}(S^n)$ for $n \geq 3$.

Homology of Eilenberg-Mac Lane Spaces. Since Eilenberg-Mac Lane spaces are naturally targets, it is difficult and interesting to study maps out of them. In particular, we might want to study $\widetilde{H}_*(K(G, n))$. Interestingly, the homology of Eilenberg-Mac Lane spaces is intimately tied to the homotopy groups of spheres.

Problem 20.18 Let $i : S^n \to K(\mathbb{Z}, n)$ be a generator for $\pi_n(K(\mathbb{Z}, n)) = \mathbb{Z}$, and let F be the fiber of i.

(a) Show that $H_{n+1}(K(\mathbb{Z}, n); \mathbb{Z}) \cong \pi_{n+1}(S^{n+1})$.

Hint Show they are both isomorphic to a third group.

(b) Determine $H_{n+1}(K(\mathbb{Z}, n); \mathbb{Z})$.

20.3 Cohomology and Lusternik-Schnirelmann Category

For any space X and any ring R, we define the R-**cup length** of X to be the largest integer n for which there is a nontrivial n-fold cup product in $\widetilde{H}^*(X;R)$. We'll denote it $\operatorname{cup}_R(X)$. The fact that $\operatorname{cup}_R(X) \leq \operatorname{cat}(X)$ is one of the most important and most useful estimates in the theory of L-S category.

Theorem 207 *For any space X, $\operatorname{cup}_R(X) \leq \operatorname{cat}(X)$.*

Proposition 208 *Let X be a space with $\operatorname{cat}(X) = n$, and let $m > n$.*

(a) Let $u_0, u_1, \ldots, u_m$ be cohomology classes with $u_k \in \widetilde{H}^{n_k}(X;G_k)$. Show that the cup product of these classes is given by the diagram

$$\begin{tikzcd}
X \arrow[r, "\bar{\Delta}_{m+1}"] \arrow[drr, "u_0\cdots u_m"'] & X \wedge \cdots \wedge X \arrow[r, "u_0\wedge\cdots\wedge u_m"] & K(R,n_0) \wedge \cdots \wedge K(R,n_m) \arrow[d, "c"] \\
 & & K(R, n_0 + \cdots + n_m).
\end{tikzcd}$$

(b) Show that if $u_0, u_1, \ldots, u_m$ are cohomology classes with $u_k \in \widetilde{H}^{n_k}(\Sigma X;R)$, then

$$u_0 \cdot u_1 \cdots u_m = 0 \in \widetilde{H}^{n_0+\cdots+n_m}(X;R).$$

Problem 20.19 Using the Whitehead definition of category and cellular approximation, show that if X is n-dimensional and $(k-1)$-connected, then $\operatorname{cat}(X) \leq \frac{n}{k}$.

Problem 20.20 Determine the L-S category of the following spaces:

(a) $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$.

(b) $\mathbb{F}\mathrm{P}^n/\mathbb{F}\mathrm{P}^{k-1}$

20.4 Moore Spaces

We show that Moore spaces can be characterized by their homology. [2]

Problem 20.21

(a) Determine $H_*(M_n(G);\mathbb{Z})$.

(b) Show that if N is any simply-connected CW complex with $H_*(N;\mathbb{Z}) \cong H_*(M_n(G);\mathbb{Z})$, then $N \simeq M_n(G)$.

---

[2] Eventually: determine the groups $[M_n(\mathbb{Z}/a), M_n(\mathbb{Z}/b)]$, which is especially fun in the case $n = 2$.

20.5 Homology Decompositions

We think of CW decompositions and Postnikov decompositions as being roughly dual to one another. But there is a significant difference: the Postnikov analysis of a space attaches (or removes) one group at a time, dimension-by-dimension. But in our CW constructions, we construct each group in two steps: first we have a map which is surjective on the homotopy (or homology) groups, and then we attach more disks to render it bijective. We can actually do a domain-type construction with works group-by-group, provided X is simply-connected.

An n-**homology approximation** is a map $X(n) \to X$ which induces an isomorphism on homology for $k \leq n$, and $H_k(X(n);\mathbb{Z}) = 0$ for $k > n$. A **homology decomposition** of a space X is a sequence of homology approximations

$$X(1) \to X(2) \to \cdots .$$

Problem 20.22 Show that if X is simply-connected and it has a homology decomposition, then it is the homotopy colimit of the decomposition.

Theorem 209 *If X is simply-connected, then X has a homology decomposition.*

Problem 20.23 Let X be $(k-1)$-connected, with $k \geq 2$.

(a) Show that X has a k-homology approximation.

(b) Suppose inductively that X has an n-homology approximation $X(n) \to X$, and let F be its fiber.

(c) Show that F is $(n-1)$-connected, so that it has an n-homology approximation $M \to F$.

(d) Extend the square

$$\begin{tikzcd}
M \arrow[r, equal] \arrow[d] & M \arrow[d] \\
F \arrow[r] & X(n),
\end{tikzcd}$$

by taking cofibers, and use the resulting diagram to prove that X has an $(n+1)$-homology approximation.

Corollary 210 *If X is simply-connected and $\widetilde{H}^n(X;G) = 0$ for all G, then X has a CW decomposition with no n-cells.*

Problem 20.24 Prove Corollary 210.

Part V

Appendices

Appendix A

Diagram Equivalences of Maps

Some of our proofs[1] have rested on certain very technical results about the special role played by fibrations and cofibrations in the study of diagram homotopy equivalences. These results are best proved in the course of a more general study of the homotopy theory of categories of maps. Since such a study must be very much more detail-laden than the study of the ordinary homotopy theory of spaces, we have chosen to present a brief introduction to the homotopy theory of mapping categories here.

The following discussion rests entirely on the formal properties of fibrations and cofibrations, and as such, it is entirely dualizable. We therefore restrict our attention to one side or the other, leaving the dual proof to the reader's imagination.

A.1 The Main Theorem

We collect here, for convenience, the main theorem of this chapter and an important corollary that is frequently useful.

Theorem 211 *If $(f, g) : \alpha \simeq \beta$ is pointwise homotopy equivalence*

$$\begin{tikzcd}
A \arrow[r, "f", "\simeq"'] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
X \arrow[r, "g", "\simeq"'] & Y
\end{tikzcd}$$

---

[1] Specifically, the proofs of Theorem **??**, Theorem **??** and Theorem 126.

in map($\mathcal{T}$), *then*

- (a) *if α* and *β are cofibrations, then*
  - i. *(f, g) is a homotopy equivalence in* map($\mathcal{T}$), *and further,*
  - ii. *if we are given a homotopy inverse $\bar{f}$ for f and homotopies $H_A : \bar{f} \circ f \simeq \mathrm{id}_A$ and $K_B : f \circ \bar{f} \simeq \mathrm{id}_B$, then the remaining map $\bar{g}$ and homotopies $H_X : \bar{g} \circ g \simeq \mathrm{id}_X$ and $K_Y : g \circ \bar{g} \simeq \mathrm{id}_Y$ can be found so that*

$$(H_A, H_X) : (\bar{f}, \bar{g}) \circ (f, g) \simeq (\mathrm{id}_A, \mathrm{id}_X)$$

and

$$(K_A, K_X) : (f, g) \circ (\bar{f}, \bar{g}) \simeq (\mathrm{id}_B, \mathrm{id}_Y).$$

- (b) *if α* and *β are fibrations, then*
  - A. *(f, g) is a homotopy equivalence in* map($\mathcal{T}$), *and further,*
  - B. *if we are given a homotopy inverse $\bar{g}$ for g and homotopies $H_X : \bar{g} \circ g \simeq \mathrm{id}_X$ and $K_Y : g \circ \bar{g} \simeq \mathrm{id}_Y$, then the remaining map $\bar{f}$ and homotopies $H_A : \bar{f} \circ f \simeq \mathrm{id}_A$ and $K_B : f \circ \bar{f} \simeq \mathrm{id}_B$ can be found so that*

$$(H_A, H_X) : (\bar{f}, \bar{g}) \circ (f, g) \simeq (\mathrm{id}_A, \mathrm{id}_X)$$

and

$$(K_A, K_X) : (f, g) \circ (\bar{f}, \bar{g}) \simeq (\mathrm{id}_B, \mathrm{id}_Y).$$

The following statement, though logically a corollary of Theorem 211, is actually a lemma used in the proof of the theorem.

Corollary 212 (Lemma 215)

- (a) *If $i : A \to X$ and $j : A \to Y$ are cofibrations, and $f : X \to Y$ is a homotopy equivalence such that the diagram*

$$\begin{tikzcd}
& A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
X \arrow[rr, "f", "\simeq"'] & & Y
\end{tikzcd}$$

commutes, then it is also a homotopy equivalence in $A{\downarrow}\mathcal{T}$.

(b) If $\alpha : X \to B$ and $\beta : A \to Y$ are cofibrations, and $f : X \to Y$ is a homotopy equivalence such that the diagram

$$\begin{tikzcd}
X \arrow[rr, "f", "\simeq"'] \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\
& B &
\end{tikzcd}$$

commutes, then it is also a homotopy equivalence in $\mathcal{T}\!\downarrow\! B$.

Fibrations that are homotopy equivalent in $\mathcal{T}\!\downarrow\! B$ are frequently said to be **fiber homotopy equivalent** to one another.

Problem A.1 Let $f, g : X \to B$, and let $p : E \to B$ be a fibration. Show that the pullback fibrations $f^*E \to X$ and $g^*E \to X$ are fiber homotopy equivalent.

A.2 Homotopy in Mapping Categories

Here we develop a tiny fraction the aforementioned homotopy theory of the category map($\mathcal{T}$) of maps.

A.2.1 The Category of Maps

We will make use of the category of maps in $\mathcal{T}$, which we denote by map($\mathcal{T}$). The objects are maps $\alpha : X \to Y$ in $\mathcal{T}$, and whose morphisms $\alpha \to \beta$ are (strictly) commutative squares

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[d, "\beta"] \\
X \arrow[r, "g"] & Y
\end{tikzcd}$$

in $\mathcal{T}$; we'll write $(f, g) : \alpha \to \beta$ for such a morphism. We have studied this category before: it is the diagram category $\mathcal{T}^{\bullet\to\star}$. Consequently, we can talk about homotopies of morphisms between maps, pointwise homotopy equivalence of maps and genuine homotopy equivalence of maps.

The main theorems of this appendix concern homotopy equivalences in the category map$_*(\mathcal{T})$, so let's recall these definitions for this special case. The product of $f \in$ map($\mathcal{T}$) with the interval is the map defined by the diagram

$$\begin{tikzcd}
X \times I \arrow[r, "\mathrm{pr}_X"] \arrow[dr, "f \times I"'] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}$$

A **homotopy** of morphisms $f, g : \alpha \to \beta$ in $\mathrm{map}(\mathcal{T})$ is a morphism $H : \alpha \times I \to \beta$ such that $H \circ \mathrm{in}_0 = f$ and $H \circ \mathrm{in}_1 = g$. In the category $\mathcal{T}$, a homotopy is pair of homotopies (H_A, H_X) that are compatible in the sense that the diagram

$$\begin{tikzcd}
A \times I \arrow[r, "H_A"] \arrow[d, "\alpha \times \mathrm{id}_I"'] & B \arrow[d, "\beta"] \\
X \times I \arrow[r, "H_X"] & Y
\end{tikzcd}$$

is strictly commutative. We will refer to H_X as a **homotopy under** H_A, and to H_A and H_X together as a pair of **coherent homotopies**. A morphism $(f, g) : \alpha \to \beta$ is a pointwise homotopy equivalence if f and g are both homotopy equivalences; it is a **homotopy equivalence** in $\mathrm{map}(\mathcal{T})$ if there is $(\bar{f}, \bar{g}) : \beta \to \alpha$ and homotopies (in $\mathrm{map}(\mathcal{T})$)

$$(H_A, H_X) : (\bar{f}, \bar{g}) \circ (f, g) \simeq (\mathrm{id}_A, \mathrm{id}_X)$$

and

$$(K_A, K_X) : (f, g) \circ (\bar{f}, \bar{g}) \simeq (\mathrm{id}_B, \mathrm{id}_Y).$$

In the following problem you will establish many useful basic properties of homotopies in mapping categories.

Problem A.2 Let $(f, g) : \alpha \to \beta$, with the notation above.

(a) If $\alpha : A \to X$ is a cofibration, and $H_A : A \times I \to B$ is a homotopy from f to $\bar{f}$, then there exists a homotopy $H_X : X \times I \to Y$ from f to some other map such that (H_A, H_X) is a homotopy in $\mathrm{map}(\mathcal{T})$.

(b) If H_X is a homotopy under H_A and K_X is a homotopy under K_A, then $H_X + K_X$ is a homotopy under $H_A + K_A$.

(c) If H_X is a homotopy under H_A, and H'_X and H'_A are the results of applying the same reparametrization to both homotopies, then H'_X is a homotopy under H'_A.

(d) In particular, if STATIC_f is the static homotopy from A, and (f, g) is homotopic to (f, g') under STATIC_f and (f, g') is homotopic to (f', g'') under H_A, then (f, g) is homotopic to (f', g'') under H_A (and similarly in the reverse order).

(e) if $(f, g) \simeq (f'g')$ under H_A and $(f', g') \simeq (f'', g'')$ under $-H_A$, then $(f, g) \simeq (f'', g'')$ under STATIC_f.

(f) Let $\alpha, \beta, \gamma \in A{\downarrow}\mathcal{T}$. Suppose $\alpha \simeq \beta$ under H_A in $\mathrm{map}(\mathcal{T})$ and $\beta \simeq \gamma$ under $-H_A$ in $\mathrm{map}(\mathcal{T})$, then $\alpha \simeq \gamma$ in $A{\downarrow}\mathcal{T}$.

EXERCISE A.3 State the duals of the statements in Problem A.2.

A.2.2 Spaces Under A or Over B

We will make use of two other categories of maps: the category of **spaces under** A, which is denoted $A \downarrow \mathcal{T}$; and the category of **spaces over** B, denoted $\mathcal{T} \downarrow B$. The objects of $A \downarrow \mathcal{T}$ are maps $\alpha : A \to X$ in $\mathcal{T}$, and a morphism from $\alpha : A \to X$ to $\beta : A \to Y$ is a commutative triangle

$$\begin{tikzcd}
& A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
X \arrow[rr, "f"] & & Y.
\end{tikzcd}$$

The objects of $\mathcal{T}{\downarrow}B$ are maps $X \to B$ and morphisms are again commutative triangles. We will focus on the category $A{\downarrow}\mathcal{T}$, but everything we say has a dual statement valid in $\mathcal{T}{\downarrow}B$.

If $f : X \to Y$ is a map in $\mathcal{T}$ such that the diagram above commutes, then we'll say that $f : X \to Y$ is a **map under** A. Two maps $f, g : X \to Y$ under A are **homotopic under** A if there is a homotopy $H : f \simeq g$ such that the diagram

$$\begin{tikzcd}
& A \times I \arrow[dl, "\alpha\times\mathrm{id}"'] \arrow[dr, "\beta\circ\mathrm{pr}_1"] & \\
X \times I \arrow[rr, "H"] & & Y
\end{tikzcd}$$

is commutative. In other words, each map $f_t : X \to Y$ is required to be a map in $A{\downarrow}\mathcal{T}$ from α to β.

A pointwise homotopy equivalence is a map $f : X \to Y$ under A which happens to be a homotopy equivalence. In the presence of a notion of homotopy, we can also define homotopy equivalence in $A{\downarrow}\mathcal{T}$ using homotopies under A.

There is a forgetful functor $A{\downarrow}\mathcal{T} \to \mathcal{T}$ which takes $i : A \to X$ to X, and carries a map $f : X \to Y$ (under A) to the same map f, but forgetting the commutativity of the triangle. There is an isomorphic copy of the category $A{\downarrow}\mathcal{T}$ contained in $\mathrm{map}(\mathcal{T})$; it is the category of all maps of the form $A \to X$;

and morphisms are squares of the form

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "\alpha"'] & A \arrow[d, "\beta"] \\
X \arrow[r, "g"] & Y.
\end{tikzcd}$$

Homotopies under A correspond to those homotopies in $\mathrm{map}(\mathcal{T})$ of the special form $(\text{STATIC}_{\mathrm{id}}, H_X)$.

EXERCISE A.4 Interpret the results of Problem A.2 in terms of the categories $A{\downarrow}\mathcal{T}$ and $\mathcal{T}{\downarrow}B$.

Let $f : A \to B$. Pullback of maps defines a functor

$$\mathcal{T}{\downarrow}B \to \mathcal{T}{\downarrow}A$$

and pushout of maps defines a functor

$$A{\downarrow}\mathcal{T} \to B{\downarrow}\mathcal{T}.$$

PROBLEM A.5 Show that these functors carry homotopy equivalences to homotopy equivalences.

HINT Show it carries cylinders to cylinders.

A.2.3 Mapping Cylinders in Mapping Categories

If $(f, g) : \alpha \to \beta$ then we may form the mapping cylinders M_f and M_g. The naturality of the mapping cylinder construction in $\mathcal{T}$ gives us the factorization

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "j_A"'] \arrow[dd, "\alpha"'] & & B \arrow[dd, "\beta"] \\
& M_f \arrow[ur, "q_B"'] \arrow[dd] & \\
X \arrow[rr, "g"] \arrow[dr, "j_X"'] & & Y \\
& M_g. \arrow[ur, "q_Y"'] &
\end{tikzcd}$$

The map $M_f \to M_g$ is called the **mapping cylinder** of (f, g), and we denote it by $M_{(f,g)}$. These mapping cylinders enjoy the same formal properties of the ordinary mapping cylinders in $\mathcal{T}$.

PROBLEM A.6 Let $(f, g) : \alpha \to \beta$ in map$(\mathcal{T})$.

(a) Show that there is a pushout diagram

$$\begin{tikzcd}
\alpha \arrow[r, "{(f,g)}"] \arrow[d, "\mathrm{in}_0"'] & \beta \arrow[d] \\
\alpha \times I \arrow[r] & M_{(f,g)}.
\end{tikzcd}$$

(b) Show that there is are inclusions $\alpha \to M_{(f,g)}$ and $\beta \to M_{(f,g)}$, and a deformation retraction $M_{(f,g)} \to \beta$.

We can do a similar construction in the categories $A \downarrow \mathcal{T}$ and $\mathcal{T} \downarrow B$, but in this case, the top (or bottom) of the prism is just copies of A (or B) joined by identity arrows, and not mapping cylinders at all.

We end our discussion of mapping cylinders with an important property of the mapping cylinders of homotopy equivalences (or, more generally, mapping cylinders of maps with left homotopy inverses). Ultimately, Proposition 213 belongs in an earlier chapter, and only Proposition 214 will remain in this appendix. But for now, we present them both here.

Recall that $Q \subseteq X$ is called a **deformation retract** of X if there is a homotopy $H : \mathrm{id}_X \simeq i \circ r$, where $r : X \to Q$ is a retraction (i.e., $r|_Q = \mathrm{id}_Q$) and $i : Q \hookrightarrow X$.

Proposition 213 *If $f : X \to Y$ has a left homotopy inverse, then X is a deformation retract of the mapping cylinder M_f.*

Proof. First, if X is a deformation retract of M_f, then it is easy to see that f has a left homotopy inverse – namely, the retraction! The proof of the converse is more involved.

Let $g : Y \to X$ be a left homotopy inverse of f, so that we are given a homotopy $H : X \times I \to X$ from $g \circ f$ to id_X. Now consider the diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{in}_0"] \arrow[d, "f"'] & X \times I \arrow[d] \arrow[ddr, bend left, "H"] & \\
Y \arrow[r] \arrow[drr, bend right, "g"'] & M_f, \arrow[dr, "\overline{H}"] & \\
 & & X
\end{tikzcd}$$

which gives us a new map $\overline{H} : M_f \to X$. And composition with the inclusion $X \hookrightarrow M_f$ gives a map $J : M_f \to M_f$. Notice that $J|_X$ is the standard inclusion $x \mapsto [x, 1]$; we claim that J is homotopic to the identity on M_f by a homotopy that is constant on X.

We will define an explicit homotopy $K : J \simeq \mathrm{id}_{M_f}$. At time $s \in I$, we write $K_s([x, t])$ for $K([x, t], s)$, and the formula is

$$K_s([x,t]) = \begin{cases} [\overline{H}([x,(1-s)t+s]), t] & \text{if } g \geq 1-s \\ \\ [\overline{H}([x,(1-s)t+s]), 1-s] & \text{if } g \leq 1-s. \end{cases}$$

Now we check

$$K_s([x,1]) = [\overline{H}([x,1]), 1] = [x,1]$$

for all x and s, so when restricted to $X \subseteq M_f$, K is static at the inclusion $X \hookrightarrow M_f$. Next, we have

$$K_0([x,t]) = [\overline{H}([x,t]), 1],$$

so $K_0(M_f) \subseteq X$, and

$$K_1([x,t]) = [\overline{H}([x,1]), t] = [x,t],$$

so $K_1 = \mathrm{id}_{M_f}$. This proves that $\overleftarrow{K}$ is a deformation of M_f into X, keeping X pointwise fixed – i.e., a deformation retraction of M_f onto X. □

Now we simply observe that this result carries over unchanged to mapping cylinders in mapping categories. You can easily formulate what it means to be a deformation retract under A or over B or in $\mathrm{map}(\mathcal{T})$.

Proposition 214 *In* $\mathrm{map}(\mathcal{T})$, $\mathcal{T} \downarrow B$ *or* $A \downarrow \mathcal{T}$, *a map* $(f, g) : \alpha \to \beta$ *has a left homotopy equivalence if and only if* α *is a deformation retract of* $M_{(f,g)}$.

PROBLEM A.7 Prove Proposition 214 by verifying that the homotopy K constructed in Proposition 213 is a homotopy in $\mathrm{map}(\mathcal{T})$, $\mathcal{T} \downarrow B$ or $A \downarrow \mathcal{T}$.

A.3 Proof of Theorem 211

The proof of Theorem 211(a) is entirely formal, and hence dualizable. In view of this observation, we will focus on the results that are relevant to homotopy pushouts.

A.3.1 Homotopy Inverses Under A

The following lemma will be used to prove our main theorem of the section.

Lemma 215 *If $f : X \to Y$ is a map in $A{\downarrow}\mathcal{T}_*$ which is a homotopy equivalence in $\mathcal{T}_*$, then it is also a homotopy equivalence in $A{\downarrow}\mathcal{T}_*$.*

Before beginning the proof of Lemma 215, let's unwind the definitions and see exactly what is being asserted here. We are given a strictly commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

in which f is a homotopy equivalence and i and j are cofibrations. The lemma claims the existence of

- is a homotopy inverse $g : Y \to X$ such that the diagram

  $$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & Y \arrow[ll, "g"'] \end{tikzcd}$$

  is strictly commutative, and

- homotopies $H : g \circ f \simeq \mathrm{id}_X$ and $K : f \circ g \simeq \mathrm{id}_Y$ making the diagrams

  $$\begin{tikzcd} & A \times I \arrow[dl, "\alpha\times\mathrm{id}"'] \arrow[dr, "\alpha\circ\mathrm{pr}_1"] & \\ X \times I \arrow[rr, "H"] & & X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & A \times I \arrow[dl, "\beta\times\mathrm{id}"'] \arrow[dr, "\beta\circ\mathrm{pr}_1"] & \\ Y \times I \arrow[rr, "K"] & & Y \end{tikzcd}$$

  strictly commutative.

The proof of Lemma 215 will be accomplished in the next two problems.

PROBLEM A.8 In this problem you will reduce the proof of Lemma 215 to the special case $f \simeq \mathrm{id}_X$.

(a) Show that it suffices to prove that f has a left homotopy inverse under A.

(b) Show that there is a map $\gamma : Y \to X$ which is a left homotopy inverse for f in $\mathcal{T}$ and which has the additional property that the diagram

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X & & Y \arrow[ll, "\gamma"'] \end{tikzcd}$$

is commutative. Why are you not done with the proof?

(c) Explain why it is sufficient to prove the lemma in the special case $f \simeq \mathrm{id}_X$.

PROBLEM A.9 In this problem, we assume that $f \simeq \mathrm{id}_X$, and prove the lemma. Let $J : f \simeq \mathrm{id}_X$, and write $J_A = J \circ (i \times \mathrm{id}) : A \times I \to X$.

(a) Starting with id_X, extend J_A to obtain a new homotopy $K : X \times I \to X$ from id_X to some other map ϕ. Verify that this makes sense.

(b) Finish the proof by studying the homotopy $(\phi \circ J) - K$.

HINT Use Problem A.2(f).

A.3.2 The Proof of Theorem 211

First, let's get our heads straight: what exactly are we trying to prove?

EXERCISE A.10 Rewrite the statement of Theorem 211 in the category $\mathcal{T}$. That is, carefully explain what maps of spaces and what homotopies are asserted to exist, and how they fit together.

Now we prove the theorem.

PROBLEM A.11 Fix $\bar{f} : B \to A$ and a homotopy $H_A : \bar{f} \circ f \simeq \mathrm{id}_A$.

(a) Show that there is a homotopy inverse γ in $\mathcal{T}_*$ for g which makes the square

$$\begin{tikzcd} B \arrow[r, "\bar{f}"] \arrow[d, "\beta"'] & A \arrow[d, "\alpha"] \\ Y \arrow[r, "\gamma"] & X \end{tikzcd}$$

commute.

(b) Show that there is a homotopy under H_A from $\gamma \circ g$ to some other map ϕ.

(c) Show that ϕ is a homotopy equivalence and that $\phi \circ \alpha = \phi$ (so ϕ is a map under A).

(d) Show that ϕ has a a homotopy inverse $\bar{\phi}$ under A.

(e) Show that $\bar{\phi} \circ \gamma = g$ is a left homotopy inverse for f under H_A.

PROBLEM A.12 Verify the claim that we are free to choose the map $\bar{g}$ and the homotopies H_X and K_X.

Part VI

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